AN INVESTIGATION OF THE EFFECTS OF TRANSPORTATION LAG ON LINEAR FEEDBACK CONTROL SYSTEMS

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Submitted in partial fulfillment of
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IN
ELECTRICAL ENGINEERING

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ABSTRACT

One of the non-linearities which occur in feedback control systems is transportation lag. All of the currently popular mathematical techniques available for the analysis of these systems are severely complicated by the introduction of transportation lag. The suitability of a numerical method employing the Z-transformation to this problem was investigated, as well as the particular advantages and disadvantages of the other methods. Solutions by the numerical method were compared with solutions by other methods and the results obtained presented in tabular form. These results demonstrate the superiority of the numerical method in determining the transient response of these systems within reasonable engineering accuracy in a much shorter time.

The writers wish to express their appreciation to Dr. M. P. Pastel for his guidance during the initial stages of the work, and to Dr. G. J. Thaler for his advice and suggestions throughout the investigation and his aid in the preparation of this paper.

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1. Introduction

1.1 Purpose of the Study

Feedback control systems are seldom truly linear, but rather they are subject to the introduction of a considerable number of non-linearities. Some non-linearities may be introduced intentionally by the system designer. An example of this might be the selection of a relay servomechanism with its attendant non-linearity, when a linear system could accomplish the task just as well. On the other hand there may be no way of avoiding a non-linearity in the system design. An example of this would be the mandatory inclusion of a gear train with its attendant backlash.

A non-linearity which frequently can neither be avoided nor neglected is transportation lag.

Because of the variable influence of transportation lag in feedback control systems and its avoidance whenever possible, it has received relatively little prominence in the literature. Rather, the mathematical tools capable of analyzing systems with transportation lag have been indicated without further amplification. While it is true that the tools indicated for the analysis of systems with transportation lag will yield satisfactory results, the problem does not end there. These standard tools, while yielding accurate results with a reasonable amount of work for linear systems, require considerably more work and time when the problem is complicated by transportation lag. Consequently, a need for a better tool than those currently in use for the solution of problems involving transportation lag is indicated.

In 1954, a paper by Ragazzini and Bergen outlined a numerical

methods technique for determining the transient response of linear feedback control systems. The nature of the mathematics which the method used indicated that it might handle the non-linearity of transportation lag more conveniently than the aforementioned standard mathematical methods generally known to engineers. The purpose of this study is to evaluate the numerical methods technique of Ragazzini and Bergen in analyzing feedback control systems with transportation lag and to compare the accuracy attainable and the work involved in using this method with the accuracy attainable and work involved in using more popular methods.

In doing so, it will be necessary to describe briefly the common analytical and graphical methods available for solving linear feedback systems. The increased difficulty and complexity introduced by the transportation lag into these methods will be demonstrated. Finally, the application of the numerical method to the same problem will be presented.

1.2 Summary of the Results and Conclusions

The numerical methods technique was compared with the root locus, Bode diagram, power series expansion, and second order Pade approximation approaches to the problem of system analysis. The Bode diagram proved to be the speediest method of all for determining system stability, but this method yields no other specific information regarding the overall system performance. For the full transient response the numerical methods technique proved to be faster and more accurate than the second order Pade approximation. The root locus method, while retaining its potential for good accuracy, required a prohibitive amount of time and tedium to be considered a practical working tool for problems involving

transportation lag. The power series expansion approach to the problem proved to be totally impractical for other than extremely simple systems.

In short, the numerical methods technique is the fastest and most accurate means of determining the transient response of a feedback control system with transportation lag. Moreover, the method can be understood and applied by engineers and students whose background includes only a first course in linear servomechanisms. In the solution of the problem by the numerical methods technique, no more complicated computing device than the desk calculator is required.

In all cases, solutions determined by the various methods were compared with the transient response obtained from programming the Fourth Order Pade Approximation of the Laplace shift operator on a ten channel Donner analog computer. This result was the most correct obtainable under the conditions of time and equipment available for the study.

2. Description of the System

The feedback control system to be studied is shown in block form in figure 2-1. In order to carry out the goals of this thesis, a number of requirements were set for this system.

A frequency response diagram crossover of one radian per second for all variations of the system was desired. For this reason, the value of K in block one will always be equal to one.

The forward loop transfer function must always be low pass in nature. For this reason the following values of G(s) were selected in block two. These values also give us convenient first, second, and third order systems to investigate.

$$G(s) = 1 \tag{2.1}$$

$$G(s) = \frac{1}{0.1s + 1} \tag{2.2}$$

$$G(s) = \frac{1}{(0.1s+1)(0.05s+1)} \tag{2.3}$$

Block three contains the transportation lag. The nature of this block and its analytic representation will be discussed in detail in section 4.

The system will use unity feedback.

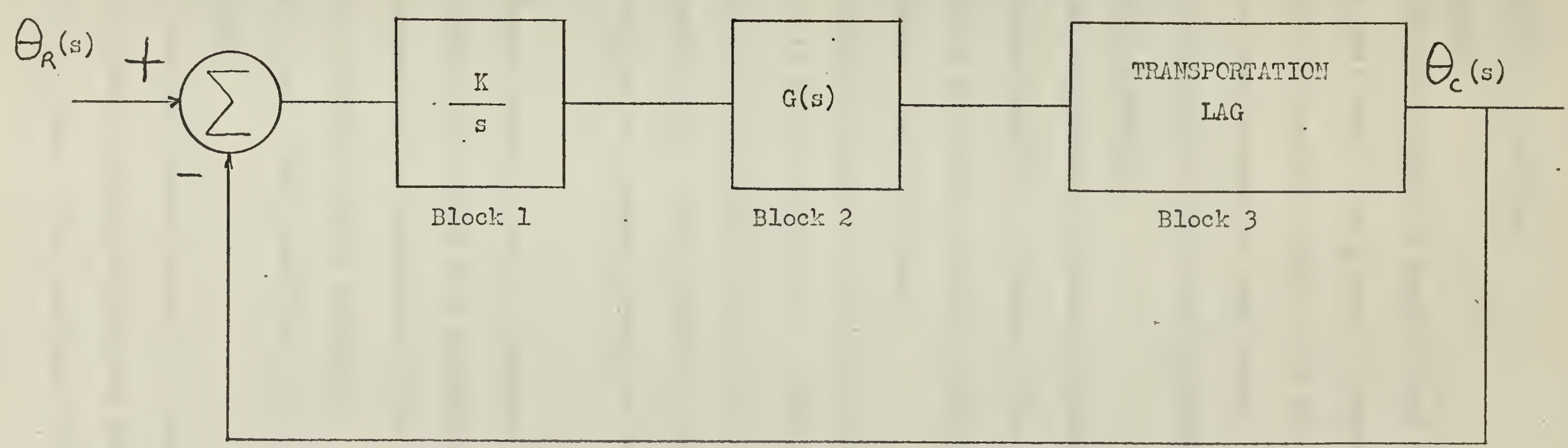


Figure 2-1. Generalized Block Diagram of Basic System.

3. Linear System Analysis

3.1 Discussion

Various methods of analysis and solution of linear feedback systems are available. Two graphical methods currently in use supply the necessary information required for the design and analysis of these systems. For the purposes of this study, the transient, or time varying response characteristics, are more important.

3.2 Frequency Response

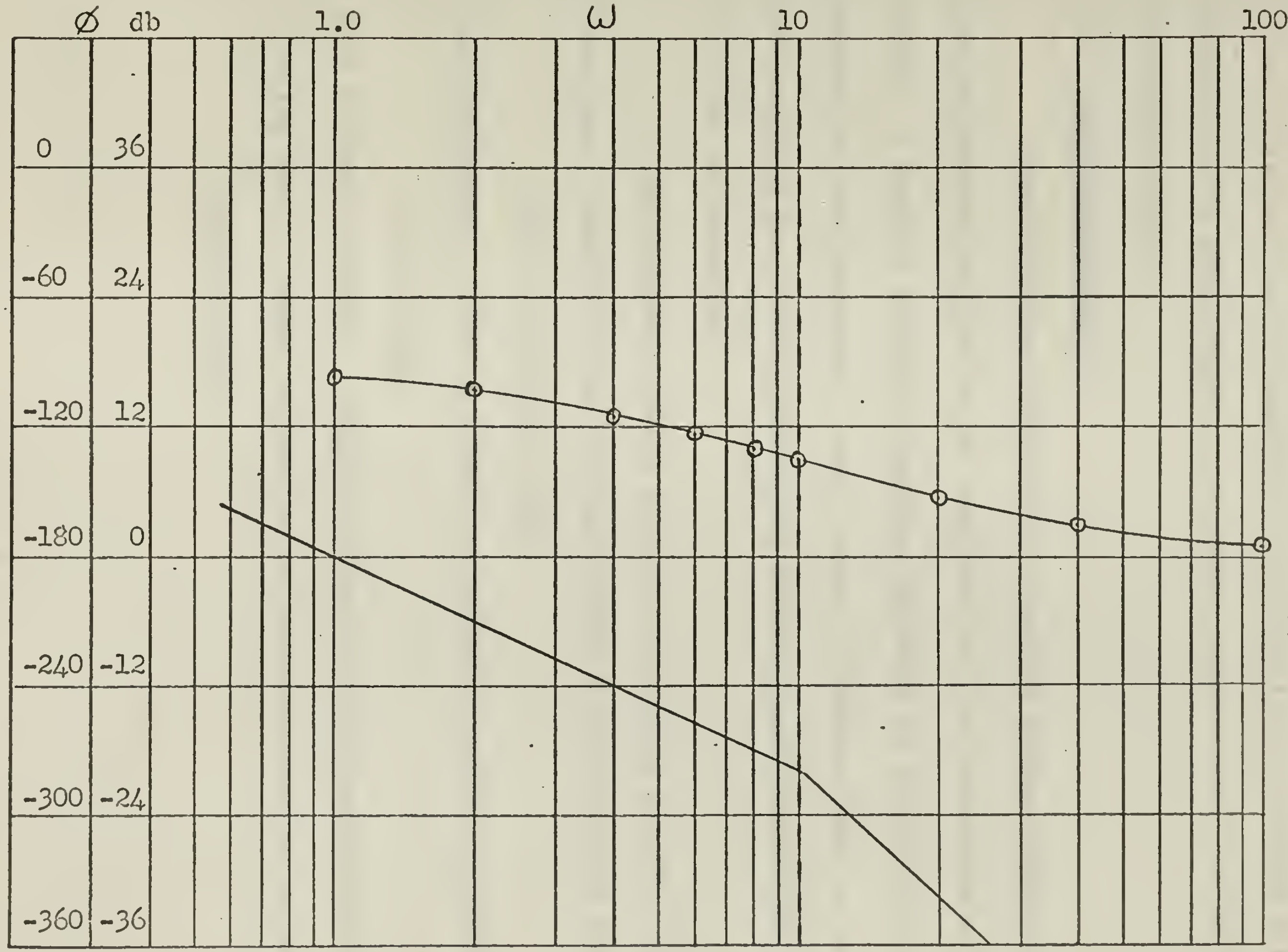
The Bode network therorems presented a new approach to the standard Nyquist stability criterion. One of the great values of the Bode diagram is the simplicity with which approximate results of sufficient accuracy may be obtained.

The Bode diagram for the linear second order system described in section 2 is shown in figure 3-1. Phase margin and gain margin values are indicated, and reasonable predictions of system performance may be made from these values. This graphical presentation of the open loop frequency response is plotted knowing the location of the open loop poles and zeros.

System stability may be inferred from the values of gain margin and phase margin on the open loop frequency response. Information from the Bode diagram may be transferred to the Nichols chart, and the closed loop frequency response may be obtained. Information from the Nichols chart should agree closely with that from a basic Nyquist plot.

3.3 Root Locus

A graphical method exists for determining the locus of the roots of the closed loop system from the location of the poles and zeros of the open loop system. The root locus method essentially demonstrates



$$G(s) = \frac{1}{s(.1s+1)}$$

Phase Margin = 82°

Figure 3-1. Frequency Response Diagram of Undelayed Second Order System

the transient response characteristics of the system rather than the frequency response.

The root locus plot for the linear second order system described in section 2 is shown in figure 3-2. Since this method indicates directly the effect on the closed loop response of the system gain from zero to infinity, it is obvious that this system is stable for all values of K. This may also be inferred from inspection of figure 3-1, which shows that the phase angle curve always remains above the 180 degree phase angle line.

3.4 Transient Response

The transient response of a control system contains nearly all the parameters required by the engineer for the evaluation of its performance. A typical transient response is shown in figure 3-3 and the vital parameters are indicated. From a study of these values, an estimate of the worth of the control system may be made, and the need for compensation may be determined.

The parameters which will be of interest are the following:

Rise Time - Length of time for the system to first reach a point within 5% of the desired value.

Peak Overshoot - Value in percent by which the system overshoots the desired value.

Time to Peak Overshoot - Length of time to reach peak overshoot point.

Settling Time - Length of time for the system to settle within 5% of the desired value.

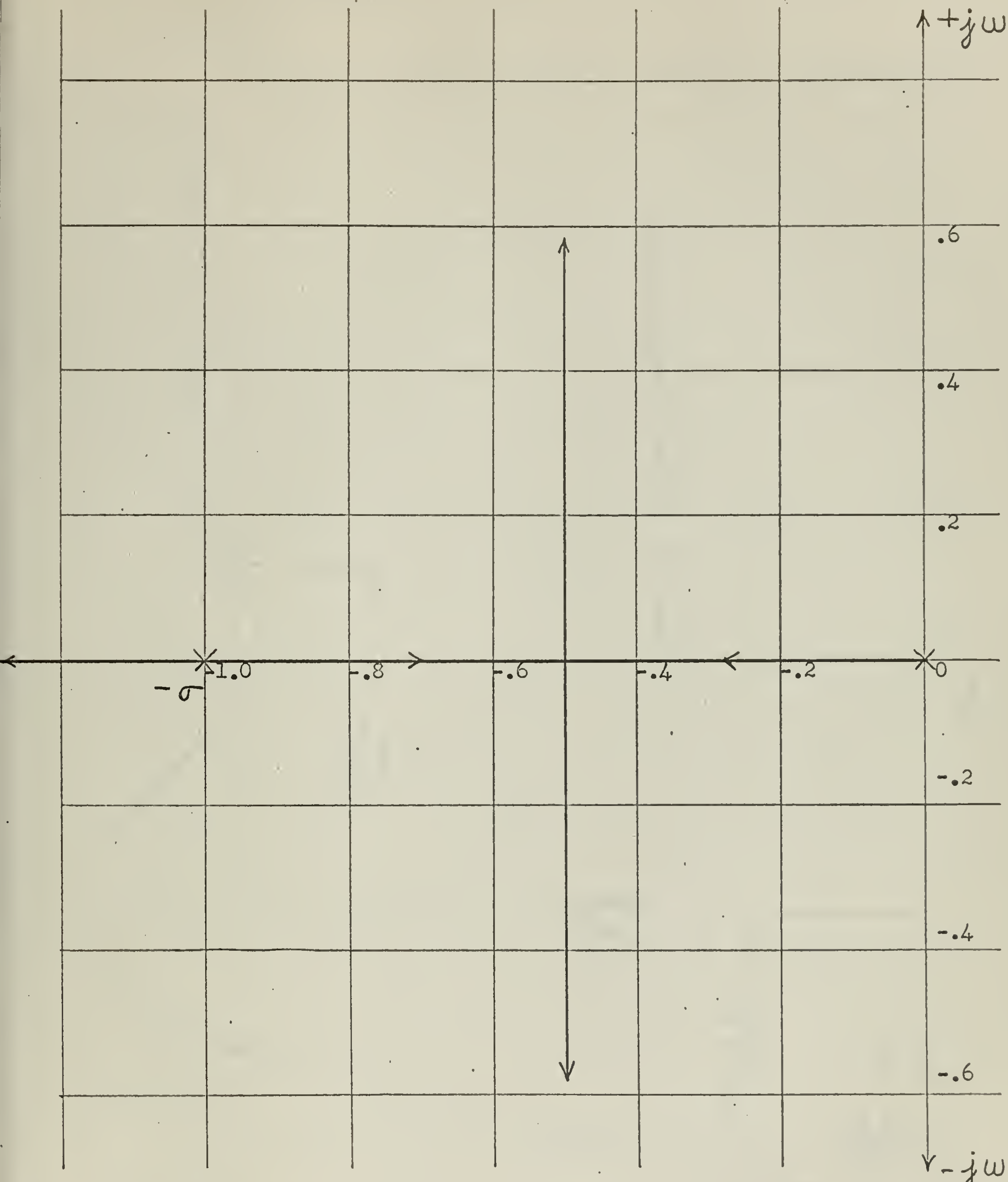


Figure 3-2. Non-Dimensionalized Root Locus of Linear Second Order System

$$G(s) = \frac{10}{s(s+10)}$$

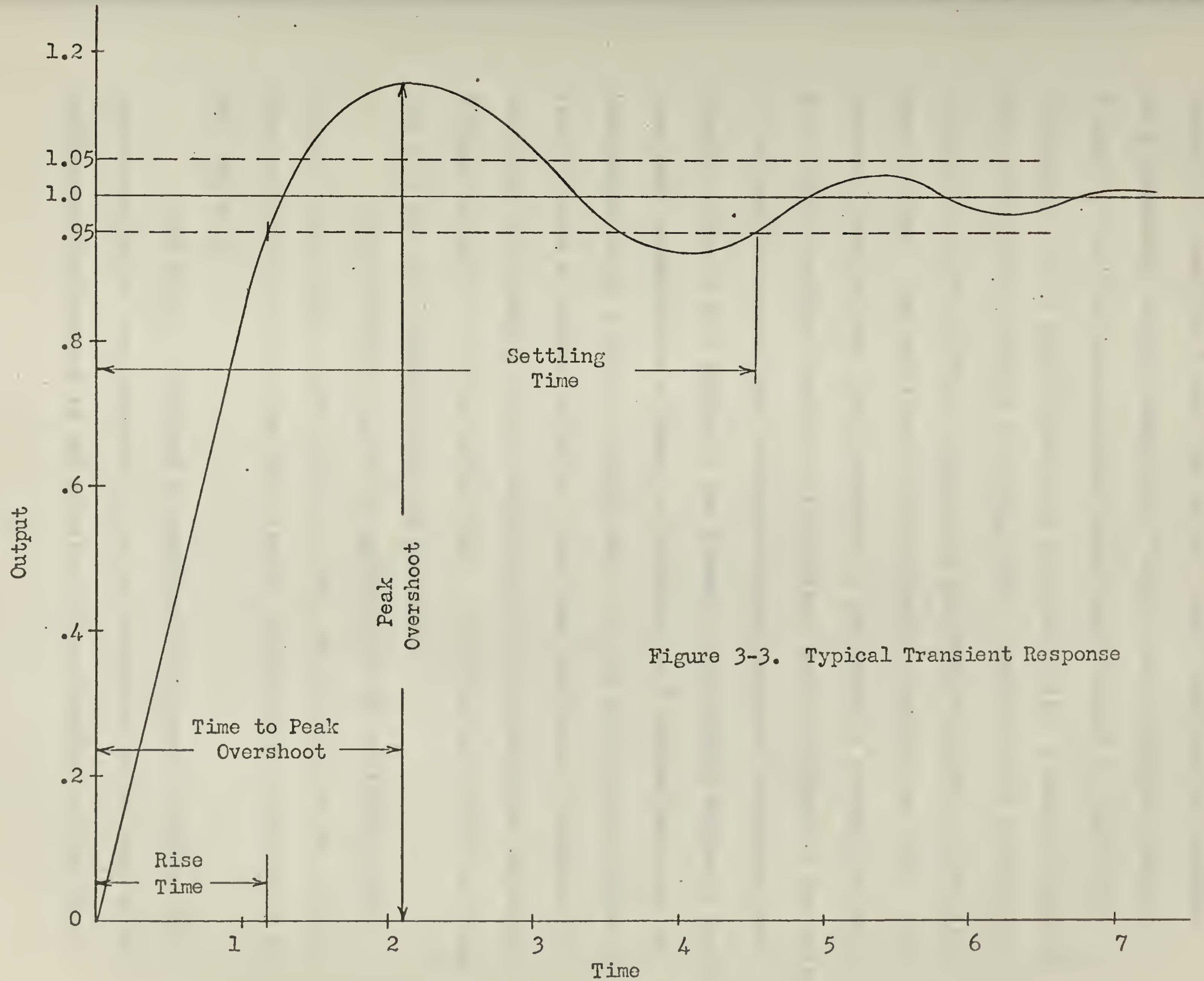


Figure 3-3. Typical Transient Response

4. Transportation Lag

4.1 Discussion

"Transportation lag" is a term frequently used in the process industry to describe a pure time delay. Other terms used to describe this phenomenon include "dead time", "delay time", and "true time lag". Figure 4-1 outlines an arrangement which might be used to control the thickness of steel plate produced by a rolling mill. A sensor indicates the actual plate thickness in voltage form. The difference between its output and another voltage representing the desired thickness forms the error voltage. The amplified error ultimately positions the rolls. The transfer function from plate thickness at the sensor X through the amplifier, motor, and gear train to roll position, hence thickness at the rolls Y, differs in no sense from ordinary transfer functions considered previously. From Y to X however a new element is introduced because a finite time must elapse before a change in thickness at Y reaches the point of measurement at X, d units of length away. If the steel plate proceeds to the left with a constant velocity v, any time function of thickness y(t) resulting from changes in roll position will be reproduced at the sensor without distortion d/v time units later. It is precisely this time quantum that the term transportation lag describes.

Transportation lag may be expressed by the following mathematical relationship: Given f(t) for $t \geq 0$, the function f(t-T) differs from the function f(t) by the dead time T, where f(t-T) is defined to be zero for $t < T$.

The setup of problems on computers occasionally requires the representation of transportation lag or an expression of the form f(t-T). The brute-force approach to the solution of this problem, and one that

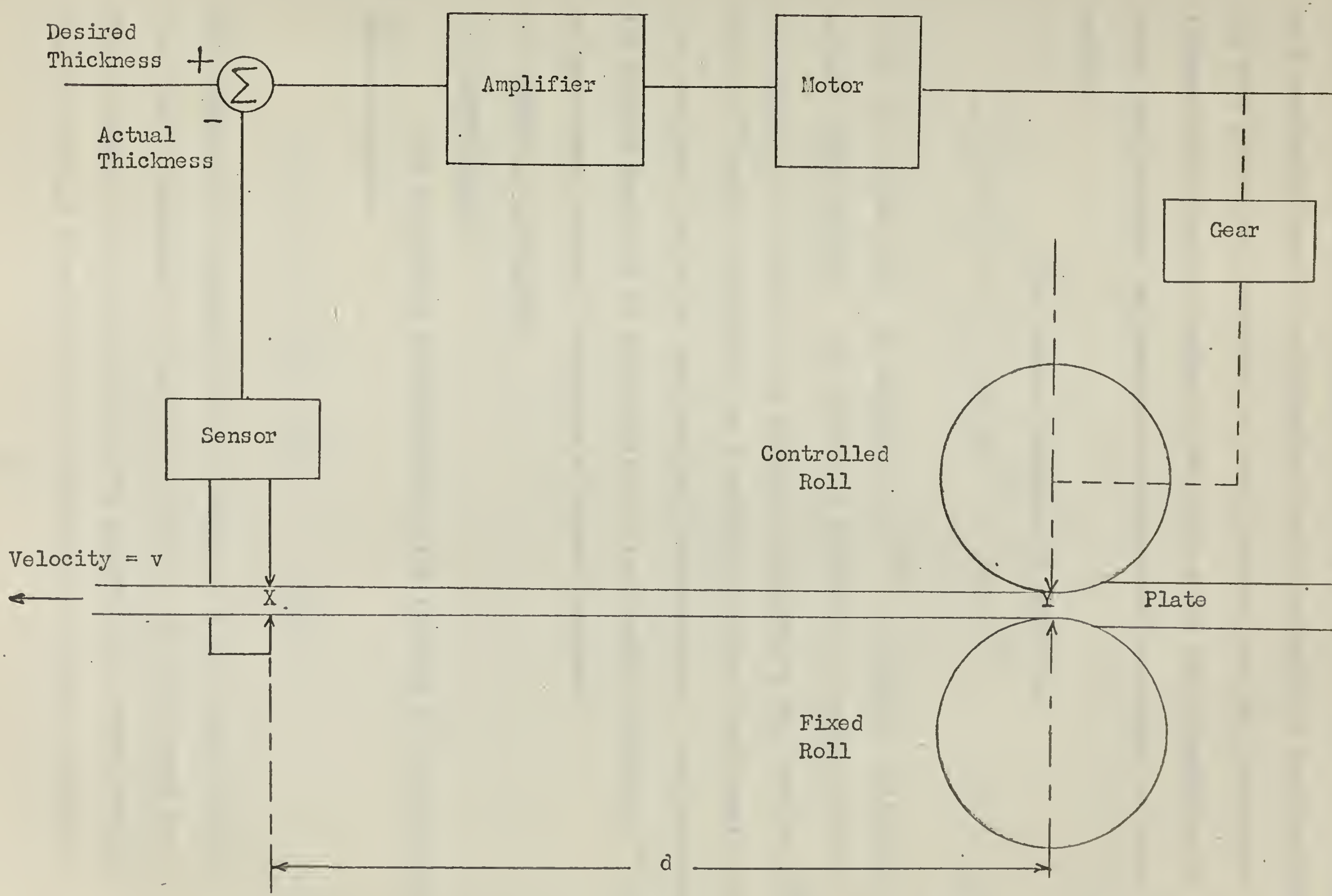


Figure 4-1. Feedback Control System for Steel Rolling Mill Application

has been used in the past, is to generate f(t) in the computer and plot f(t) vs. t on a specially equipped recording device. Schemes involving such mechanical devices to accomplish the introduction of transportation lag have proved to be workable for some problems, but the limitations imposed by the physical design of the devices and the introduction of human error in their use have made necessary the development of better methods.

The Laplace transform of f(t-T) is

$$\mathcal{L}[f(t-T)] = \mathcal{L}[f(t)]e^{-sT} = F(s)e^{-sT} \tag{4.1}$$

if f(t-T) = 0 for $t < T$. In equation 4.1 e^{-sT} is commonly called the Laplace shift operator. The problem of generating a function f(t-T) can therefore be reduced to the problem of generating the function e^{-sT}. Since e^{-sT} cannot be expressed as a rational fraction, the transfer function for dead time can be simulated only approximately. Thus the remaining methods depend on an approximation of the transfer function e^{-sT} by an approximate series or technique of network synthesis.

4.2 Methods of Simulation

4.21 Power Series

The best known approximation to the function e^x is the Taylor series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \cdot + \frac{x^n}{n!} \tag{4.2}$$

This expansion is convergent for all values of x, but unfortunately the rate of convergence is small for x large. For this reason the Taylor series is not well suited to the generation of transportation lag for problems involving high frequencies or long time constants.

This can be seen by replacing s by $j\omega$ in the expression for e^{-sT}. Moreover, in attempting to determine by longhand calculations, the transient response of other than the simplest system possible involving transportation lag, the introduction of the Taylor series approximation for e^{-sT} complicates the problem so severely as to render it useless.

4.22 <u>The Pade Approximation</u>

In section 4.1 an important theorem of Laplace transforms was stated, viz.:

$$\mathcal{L}[f(t-T)] = \mathcal{L}[f(t)]\,e^{-sT} \qquad \text{repeated 4.1}$$

Thus the transfer function of transportation lag is

$$\frac{\mathcal{L}[f(t-T)]}{\mathcal{L}[f(t)]} = e^{-sT} = \frac{E_o}{E_i} \qquad 4.3$$

Substituting $s = j\omega$ to find the frequency response gives

$$\frac{E_o}{E_i} = e^{-j\omega T} = \cos\omega T - j\sin\omega T$$

Thus $$\left|\frac{E_o}{E_i}\right| = \sqrt{\cos^2\omega T + \sin^2\omega T} = 1 \qquad 4.4$$

$$\phi = \tan^{-1}\left(-\frac{\sin\omega T}{\cos\omega T}\right) = -\omega T \qquad 4.5$$

or the frequency transfer function of transportation lag is characterized by an amplitude ratio which is unity for all frequencies and a phase angle which varies linearly with frequency over the desired range of interest.

A popular method of approximating the transfer function e^{-sT} from this point of view is the Pade approximation. It stems from the fact that the quotient of any pair of complex-conjugate polynomials has an amplitude of unity and also possesses considerable phase shift. Expressed in its basic form, the Pade approximation is

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$e^{x} = \lim_{(u+v)\to\infty} \frac{F_{u,v}(x)}{G_{u,v}(x)} \tag{4.6}$$

where

$$F_{u,v}(x) = 1 + \frac{vx}{(u+v)\,1!} + \frac{v(v-1)\,x^2}{(u+v)(u+v-1)\,2!} + \cdots$$

$$\cdots + \frac{v(v-1)\cdots 2\cdot 1\;x^{v}}{(u+v)(u+v-1)\cdots(u+1)\,v!} \tag{4.7}$$

$$G_{u,v}(x) = 1 - \frac{ux}{(v+u)\,1!} + \frac{u(u-1)\,x^2}{(v+u)(v+u-1)\,2!} + \cdots$$

$$\cdots + (-1)^{u} \frac{u(u-1)\cdots 2\cdot 1\;x^{u}}{(v+u)(v+u-1)\cdots(v+1)\,u!} \tag{4.8}$$

The convergence for this series expansion is quite rapid and often values of u and v of 2 will give good accuracy for short time lags and low frequencies. The Pade approximant of e^{-sT} for u = v = 2 is given in equation 4.10 below.

$$e^{-sT} \cong \frac{1 - \frac{2}{4}(sT) + \frac{2}{4!}(s^2T^2)}{1 + \frac{2}{4}(sT) + \frac{2}{4!}(s^2T^2)} \tag{4.9}$$

$$e^{-sT} \cong \frac{s^2T^2 - 6sT + 12}{s^2T^2 + 6sT + 12} \tag{4.10}$$

In terms of the original problem, generating f (t-T) when f(t) is known, the equation for the second-order Pade approximant can be expressed in transfer-function notation as

$$\frac{f(t-T)}{f(t)} = \frac{D^2T^2 - 6DT + 12}{D^2T^2 + 6DT + 12} \tag{4.11}$$

The justification of the validity of equation 4.11 is demonstrated as follows:

$$\mathcal{L}\left[f(t-T)\right] = F(s)\,e^{-sT} = \mathcal{L}\left[f(t)\right]e^{-sT} \qquad \text{repeated } 4.1$$

Substituting equation 4.11 in equation 4.1 gives

$$\frac{\mathcal{L}[f(t-T)]}{\mathcal{L}[f(t)]} \cong \frac{s^2T^2 - 6sT + 12}{s^2T^2 + 6sT + 12} \qquad 4.12$$

Rewriting equation 4.12 gives

$$(s^2T^2 + 6sT + 12)\mathcal{L}[f(t-T)] = (s^2T^2 - 6sT + 12)\mathcal{L}[f(t)] \qquad 4.13$$

Performing the inverse Laplace transformation on equation 4.13 gives

$$(D^2T^2 + 6DT + 12) f(t-T) = (D^2T^2 - 6DT + 12) f(t) \qquad 4.14$$

Comparison of equations 4.14 and 4.12 reveals that the equations are identical. This is, of course, a logical conclusion, as the Laplacian operator and the differential operator may be used interchangeably providing the initial conditions in a system are identically zero. This is true in the system described above.

The fourth order Pade approximant (u = v = 4) for e^{-sT} is

$$e^{-sT} \cong \frac{(sT)^4 - 20(sT)^3 + 180(sT)^2 - 840\,sT + 1680}{(sT)^4 + 20(sT)^3 + 180(sT)^2 + 840\,sT + 1680}$$

The Pade approximants may be used for either longhand or computer solutions. For use in longhand solutions the approximant to be used is merely substituted for e^{-sT}. It is worthy of note that the order of the characteristic equation is raised by the order of the approximant used to simulate e^{-sT}. If the second order approximant is used, two additional roots are introduced into the characteristic equation, and if the fourth order approximant is used, four additional roots are introduced into the characteristic equation.

The technique of programming the Donner Analog Computer for handling the transfer function of the fourth order Pade approximant

is included in Appendix Section 9.

The method of simulating transportation lag by the Pade approximants yields what may be termed optimally flat phase characteristics at the origin but does not fit the phase characteristics as well over an extended frequency range. The major drawback to the use of the Pade approximants to simulate transportation lag is their response to a step input or other input having high-frequency components, since the amplitude ratio is always unity but the phase is approximated only for low ωT products. The phase shift characteristics of certain of the Pade approximants are shown in Figure 4-2. As a result of the inability of these approximants to linearly phase shift high frequencies, the output of an approximant "rings" badly for high-frequency inputs. This behavior is illustrated in Figure 4-3 which depicts the response of the fourth order approximant to inputs of a step and ramp with unity feedback. It is noteworthy that the "ringing" is considerably less for the ramp input than for the step input. The reason, of course, is that the ramp input contains a lower percentage of high-frequency terms than does the step input.

In the solution of problems concerning feedback control systems with transportation lag, advantage may be taken of the fact that the typical system encountered is, in effect, a low pass filter. Block diagram algebra techniques are employed in the solution of these problems. Since the commutative law is applicable to block diagram algebra, the actual location of the transportation lag transfer function in the forward path is immaterial and it may be placed at the end of the path in the computer where the high frequency components have been attenuated to the greatest possible extent. Frequencies which have been attenuated 40 db. or more

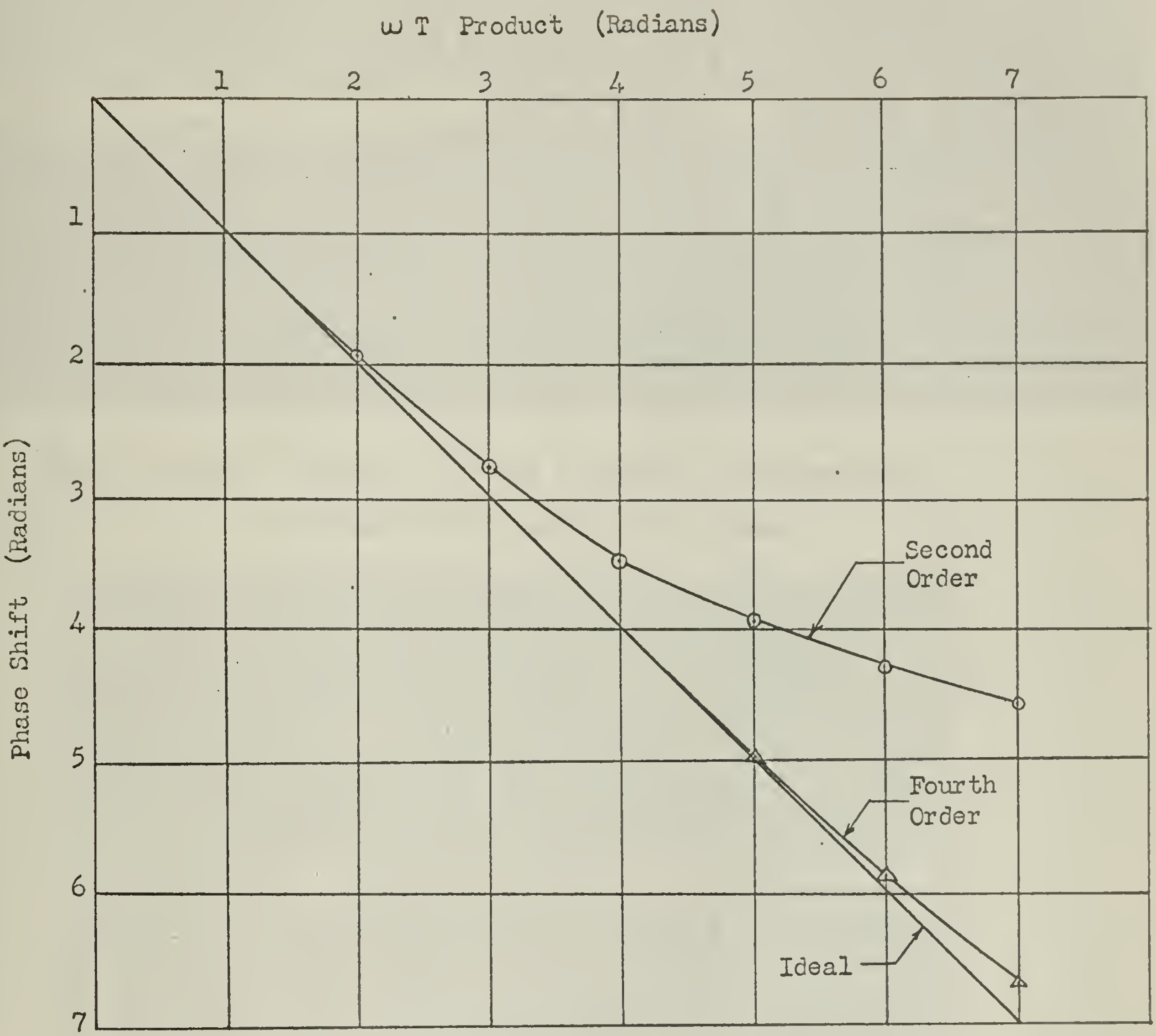


Figure 4-2. Phase Shift Characteristics of the Pade Approximation

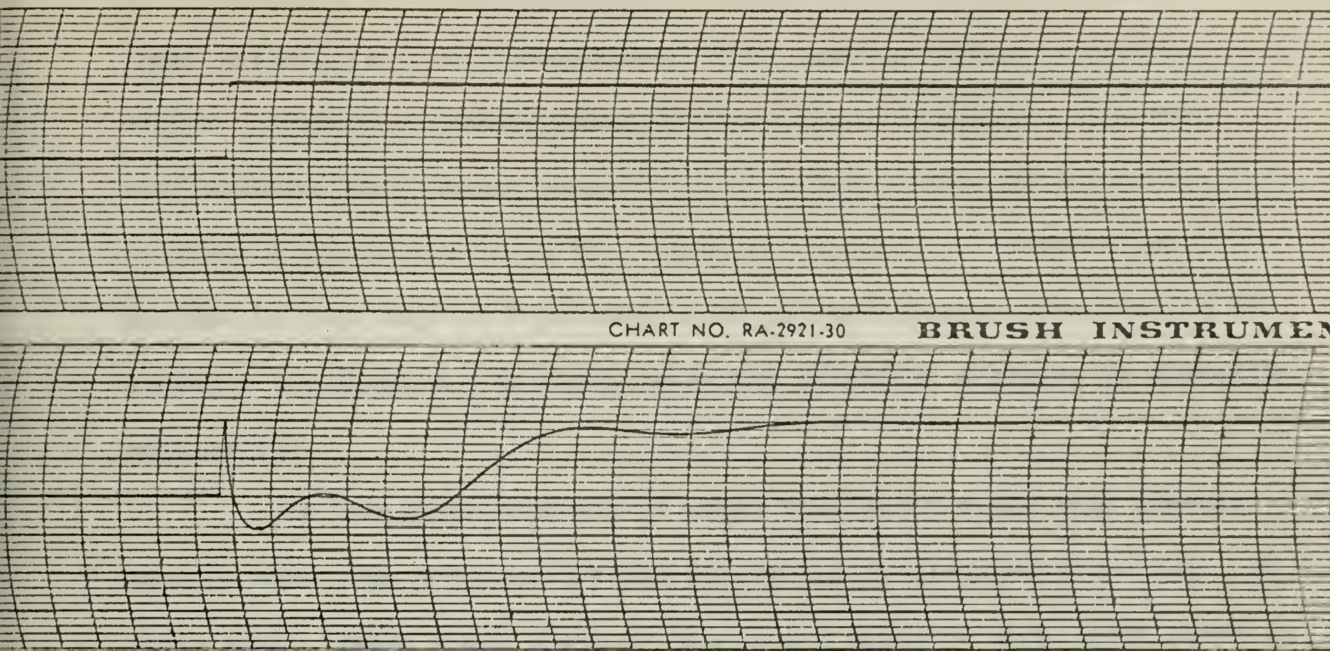


Figure 4-3(a). Response of Fourth Order Pade Approximation of 0.9 Second Time Delay to Step Input.

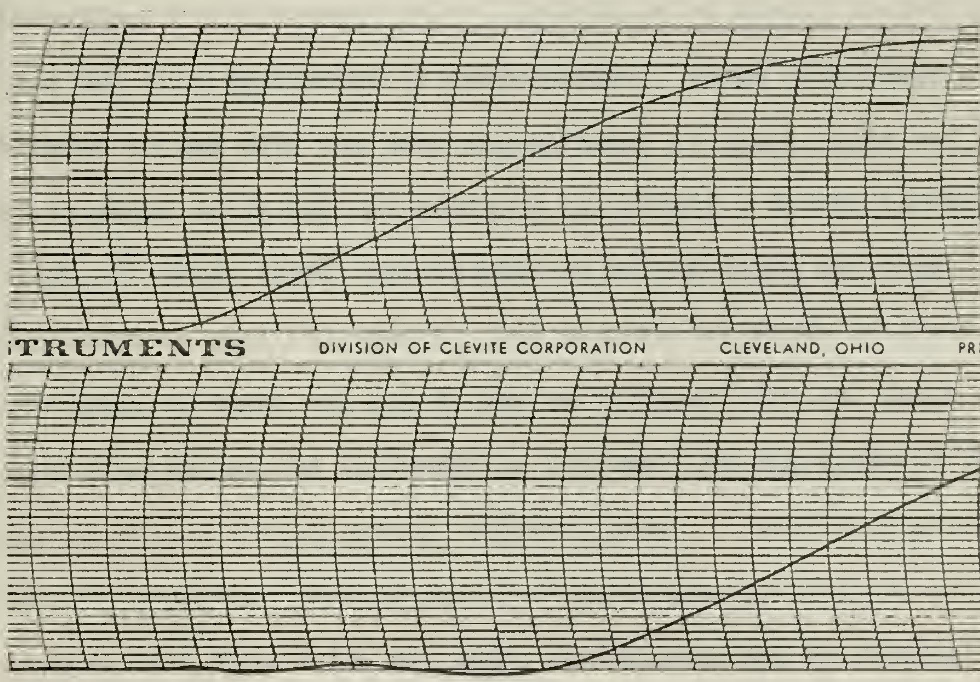


Figure 4-3(b). Response of Fourth Order Pade Approximation of 1.0 Second Time Delay to Ramp Input With Unity Feedback.

are considered to be negligible and whether they have been linearly phase shifted is of no consequence. This tends to obviate the necessity of having to linearly phase shift a high ωT product. Another consideration which makes unnecessary the linear phase shift of high ωT products in the solution of the transient response of feedback control systems is that high ωT products mean extremely large phase shifts and probable instability. In these cases the transient response is virtually meaningless.

In the great majority of cases the fourth order Pade approximant is more than adequate for the simulation of transportation lag. In cases where linear phase shift of higher ωT products than those shown in figure 4-2 is desirable, higher order Pade approximants or more complex curve fitting techniques may be used to simulate e^{-sT}. The curve fitting techniques are discussed in the literature on analog computers but the need for such approaches is questionable. These techniques were devised because it was felt the Pade approximants of sixth order and higher failed to phase shift low ωT products accurately. In the course of this study the phase shift characteristics of the sixth and eighth order Pade approximants were calculated. The results are shown in figure 4-4. On the basis of these results it may be inferred that the order of the Pade approximation may be increased to increase the magnitude of the ωT product that is linearly phase shifted without distorting the phase shift of low ωT products. Because the Pade approximation may be expressed in a general form from which any order approximant may be developed, it is felt that the use of curve fitting techniques requires a needless expenditure of effort without any tangible benefit.

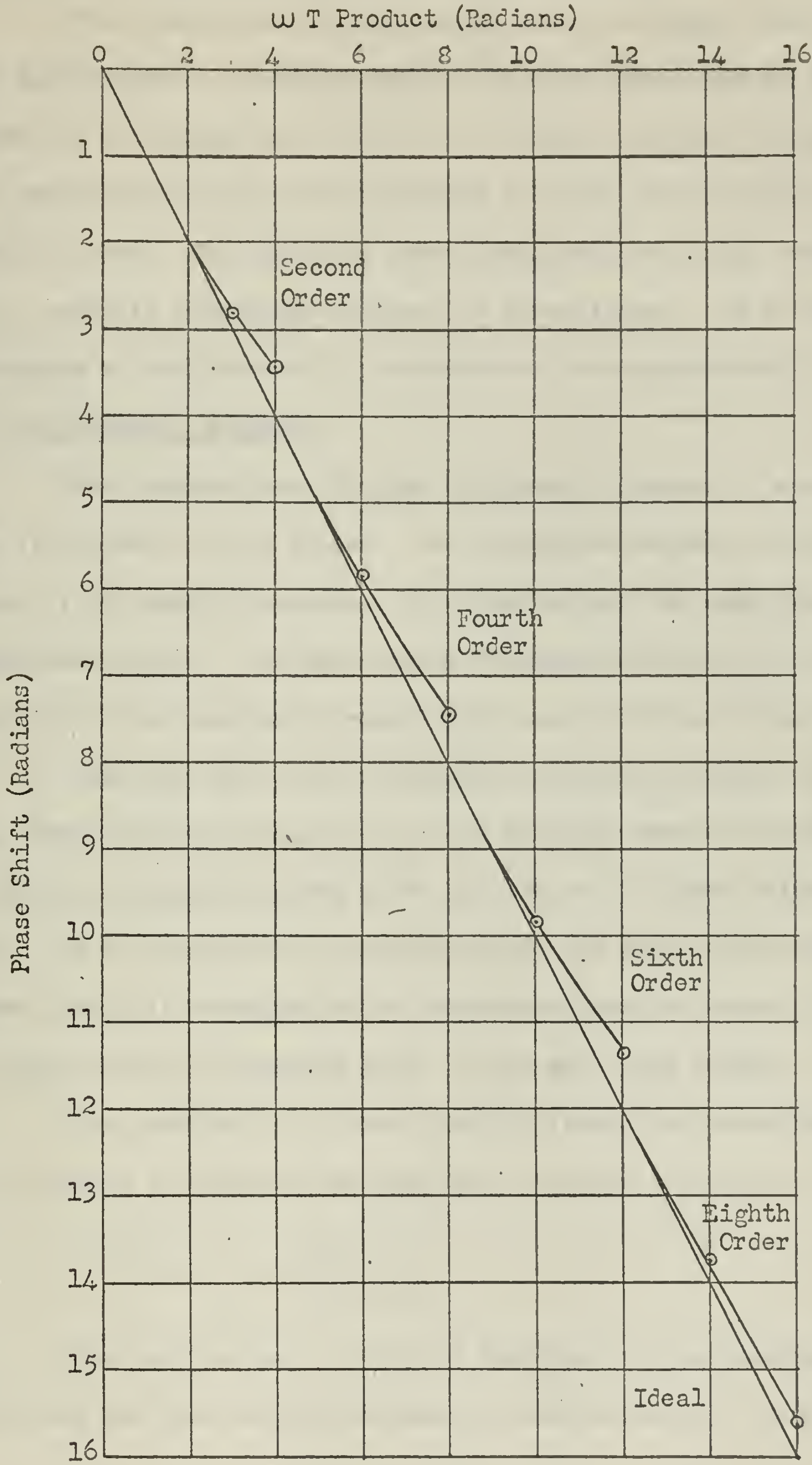


Figure 4-4. Phase Shift Characteristics of the Pade Approximation

5. Effect of Time Delay on Linear System Analysis

The addition of the time delay into the linear control system under investigation introduces several factors which tend to make the analysis of the system more difficult. Because the time delay may basically be represented by an infinite series of terms, this creates an infinite number of roots. The effect of these complications on the standard graphical analysis techniques will now be investigated. In all cases, the system with a time delay of 1.0 seconds will be demonstrated.

5.1 The Frequency Response

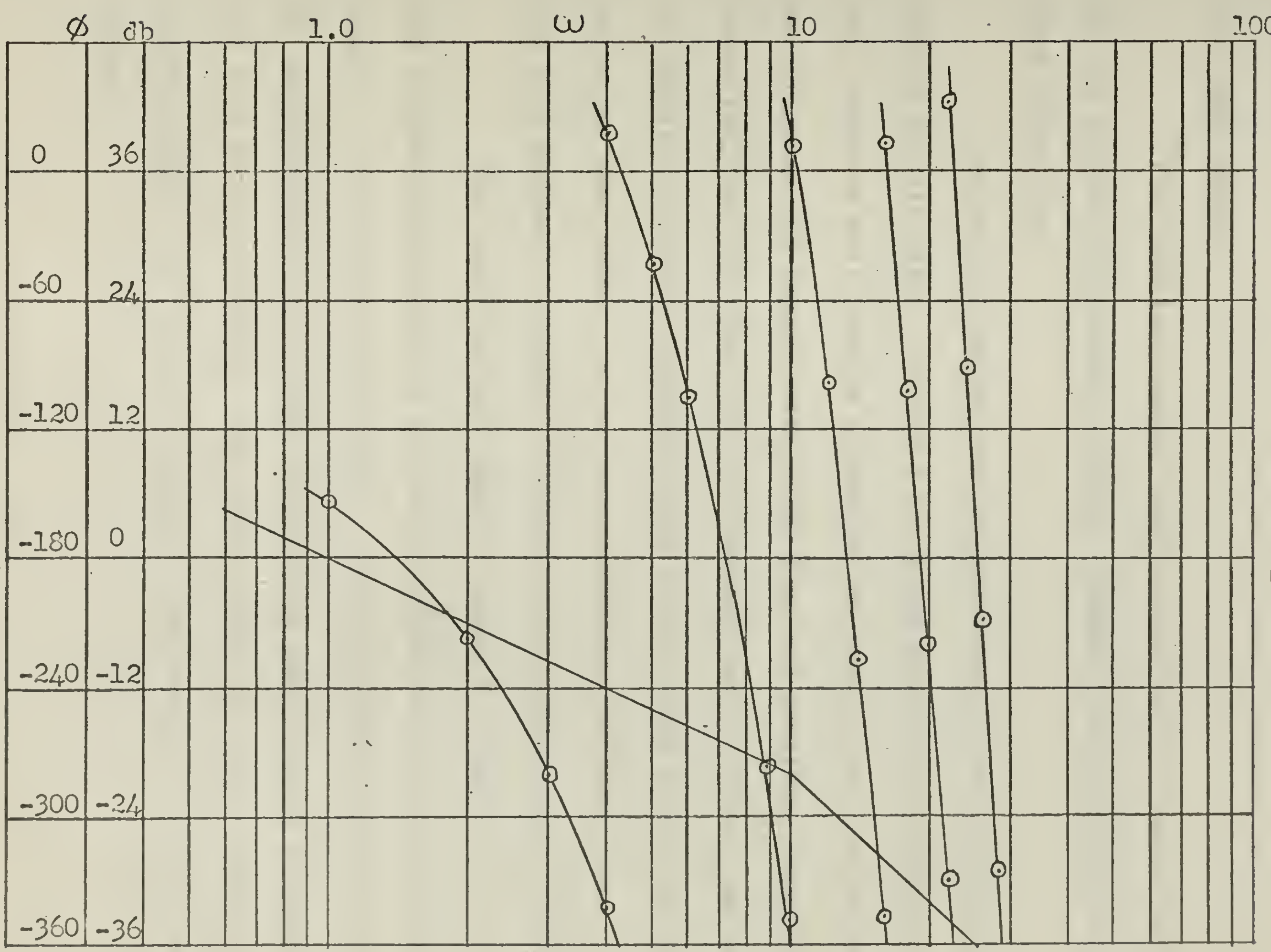
The standard Bode diagram is shown in figure 5-1 when the time delay is included in the system. The difference between this plot and figure 3-1 is readily apparent. The phase margin has been reduced by a considerable amount. The gain margin is approximately 3.3 db which compares with the undelayed system which was stable at all gains.

The time delay has introduced additional branches into the diagram. These branches appear due to the periodic nature of the time delay, and the increasing phase shift introduced at higher values of frequency. It is evident that the gain margin for each of the branches increases. This is analogous to the statement that the roots of the primary branch would go unstable first if the gain were raised.

The construction of this plot is simple and straightforward. For the system in question the open loop transfer function is

$$G_o(s) = \frac{e^{-sT}}{s(0.1s+1)} \qquad 5.1$$

The gain vs log ω curve is identical to the undelayed system curve since the time delay introduces no additional gain into the system.



$$G(s) = \frac{e^{-sT}}{s(.1s+1)}$$

Phase Margin = 28°

Gain Margin = 3.3 db

ω Stability Limit = 1.43

K Stability Limit = 1.48

Figure 5-1. Frequency Response Diagram of Second Order System With Transportation Lag of One Second.

The phase angle vs $\log \omega$ curve is drastically changed since the equation for the phase angle is

$$\phi = 90^\circ + \text{ARC TAN } 0.1\omega + \omega T_D \qquad 5.2$$

It is the presence of the additional term of the frequency times the time delay which introduces the additional phase shift and the extra branches.

5.2 Root Locus

The identical second order system with a transportation lag of 1.0 seconds also appears greatly changed when investigated by a root locus plot.

Figure 3-2 showed the undelayed second order system which is stable for all gains. Figure 5-2 has the additional construction detail which we will require to obtain the root locus. In the same manner as the minus 180 degree phase shift line was obtained, lines of constant phase shift of 170, 160, 150, 140 degrees, and so forth, may be plotted. In addition, lines of constant phase shift caused by the one second time delay are also shown. In the non-dimensionalized plot shown, the time delay will produce a complete 360 degree phase shift every 0.628 radians.

The resulting root locus is shown in figure 5-3. It is plotted by combining points of the basic system and the time delay which total to a 180 degree phase shift. It is evident that this system will go unstable with only a small increase in gain, and as a matter of fact, the gain at the stability limit just before the locus crosses the axis is approximately 1.48, the same value of gain margin obtained from the Bode diagram.

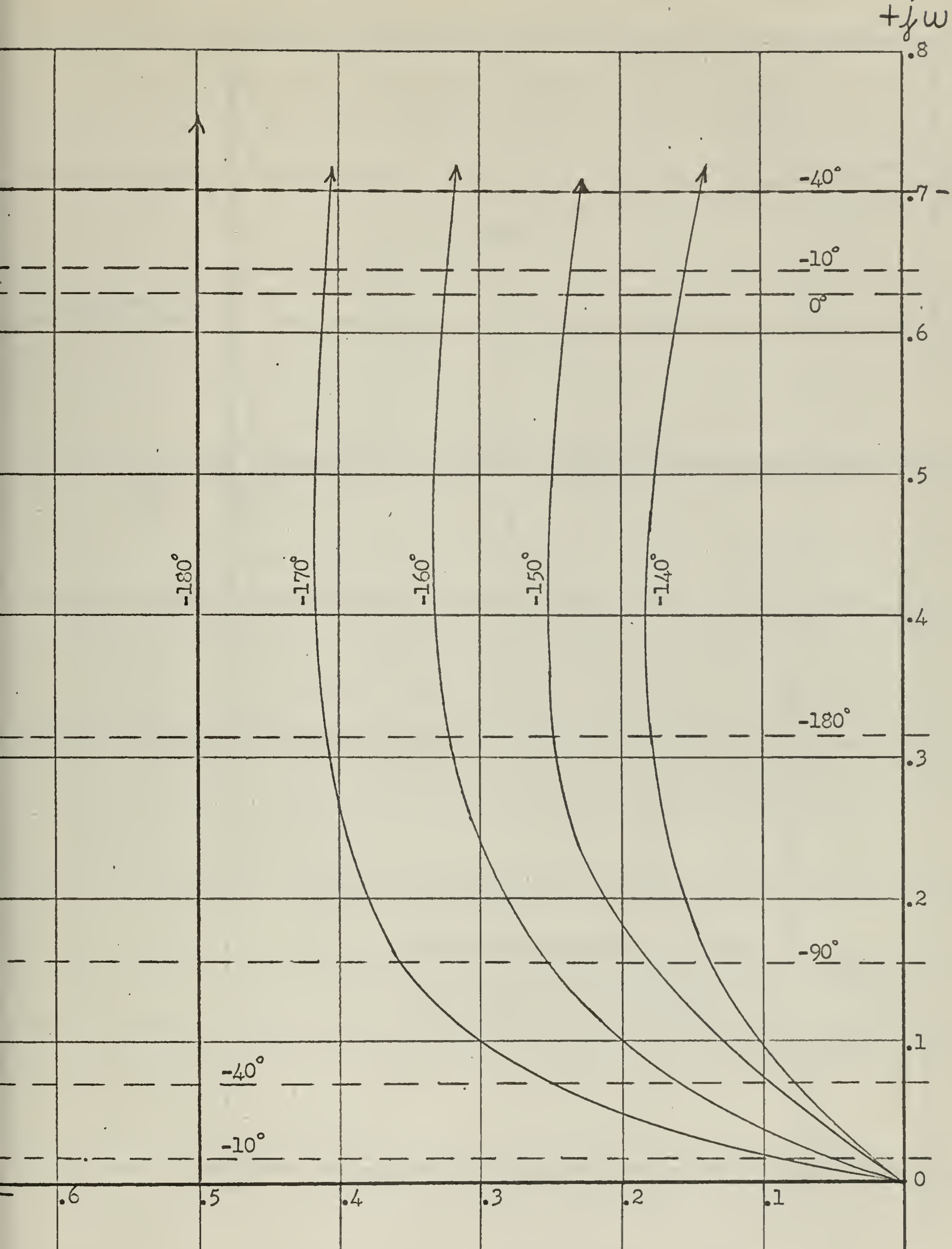
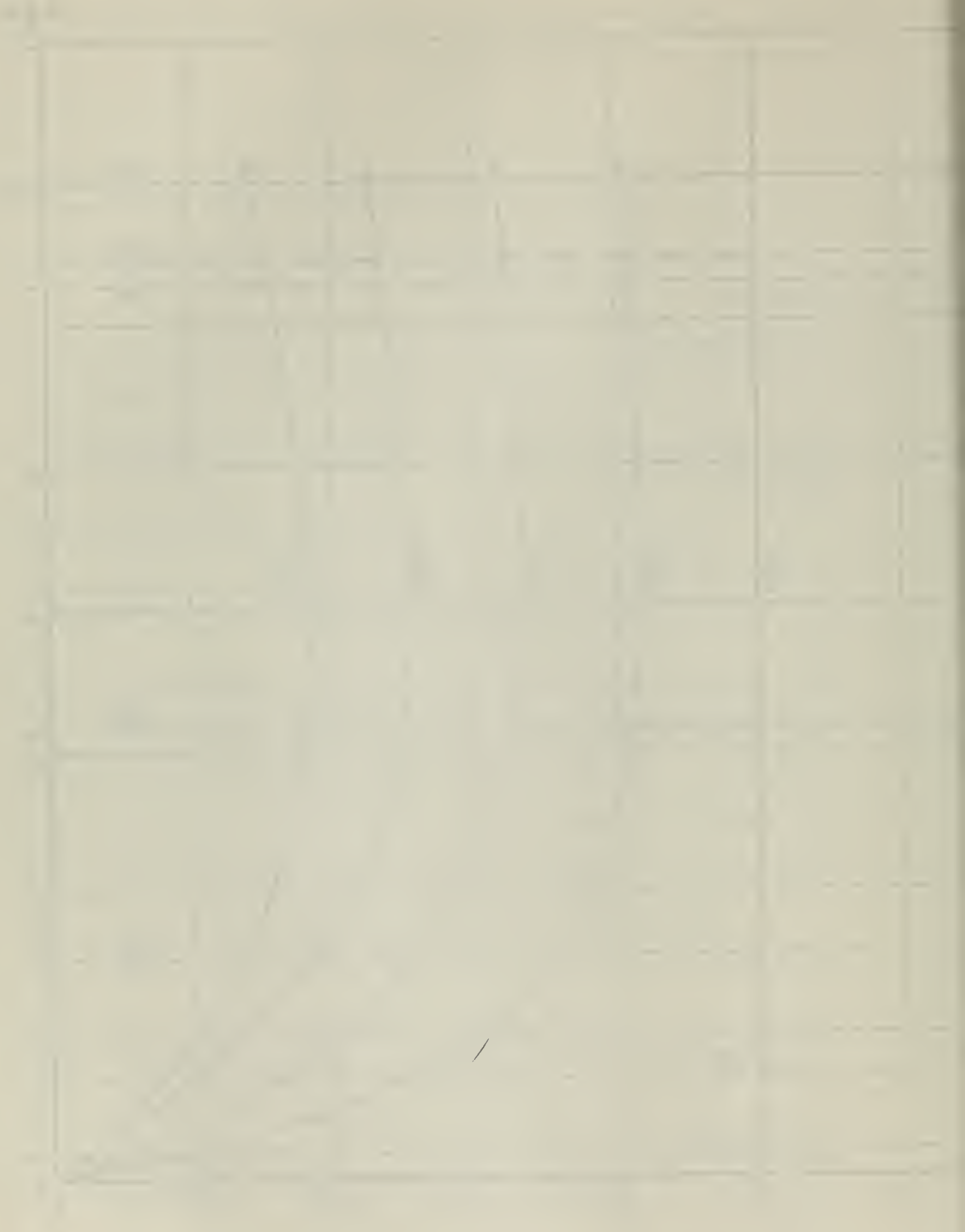


Figure 5-2. Root Locus Construction Detail



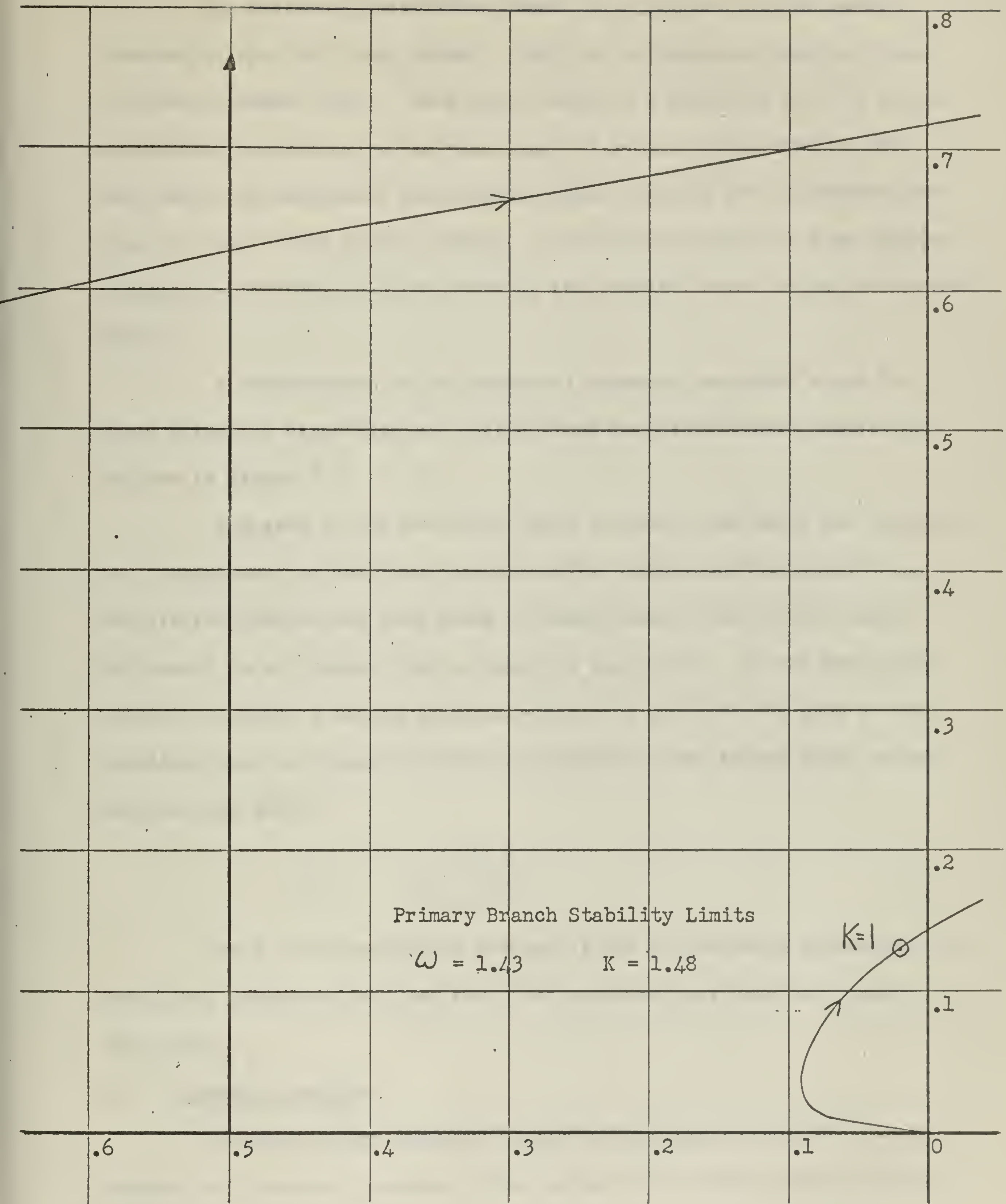


Figure 5-3. Root Locus of Second Order System With Transportation Lag of One Second.

An extremely interesting result of this plot is that upper branches of the root locus appear. Only two of these are shown although an infinite number exist. Here again they are a result of the 360 degree periodicity introduced by the time lag. It is also noted that at the same gain, the additional roots on the upper branches are attenuated more than the root on the primary branch. This substantiates the Bode diagram findings, which show that the roots on the primary branch would go unstable first.

A continuation of the graphical solution described above for other values of time delay will yield other root loci curves similar to the one in figure 5-3.

The gain at the stability limit for each time delay was calculated. The product of the time constant of the system and the gain, $K \tau_a$, was plotted against the time delay on logarithmic cross-section paper. The result is a straight line as shown in figure 5-4. It was therefore possible to write a single equation which will describe the gain at the stability limit in terms of the time constant of the second order system and the time delay.

$$K = \frac{0.148}{\tau_a T_D^{.89}} \qquad 5.3$$

Thus for second order systems if two of the above parameters are known, the maximum value that the third parameter may have can readily be calculated.

5.3 Transient Response

Neither of the graphical methods discussed here will directly produce the transient response of any system. The Bode diagram will only

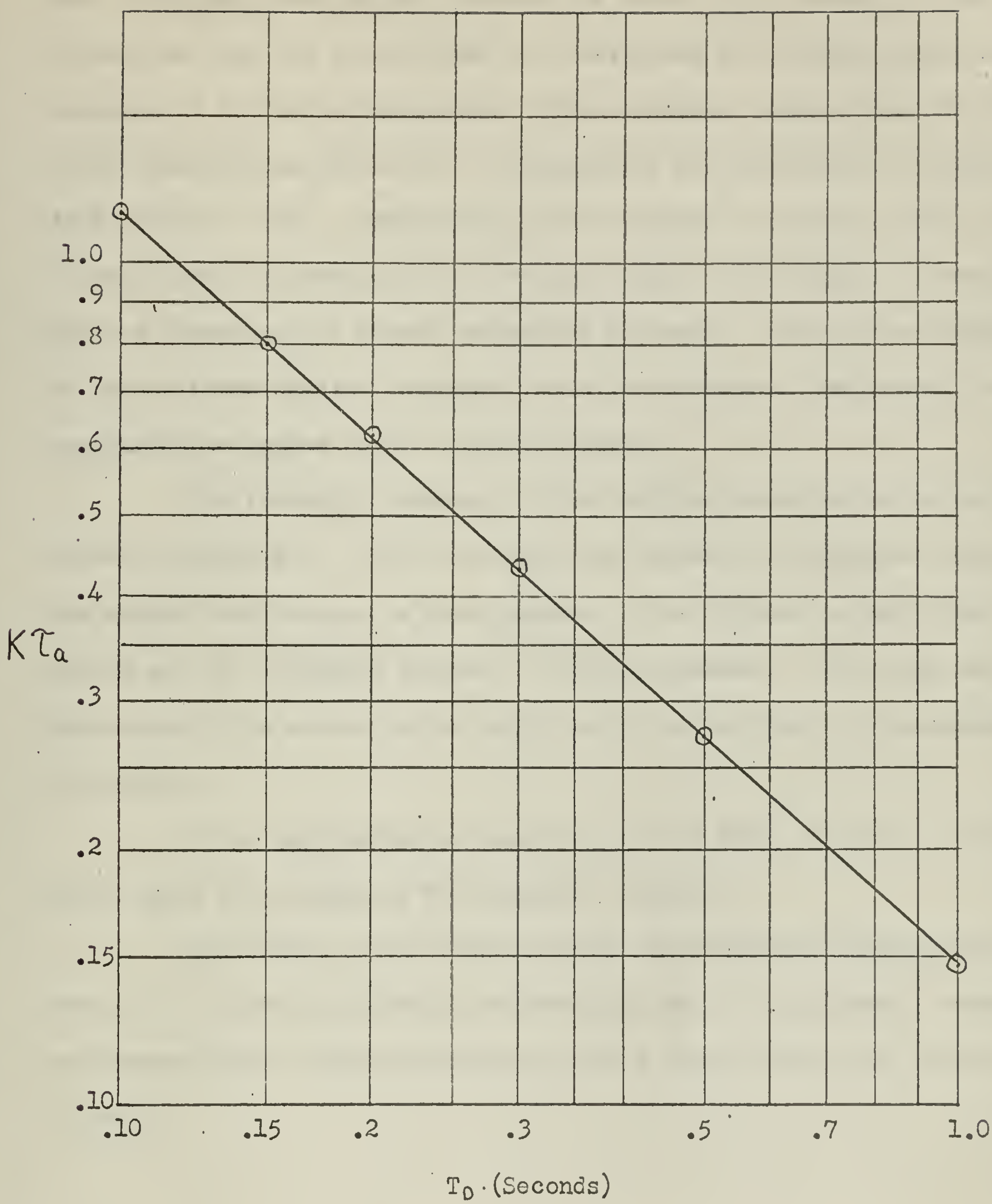


Figure 5-4. Non-Dimensionalized Gain at Stability Limit vs. Transportation Lag for Second Order System

yield values from which an approximate transient response may be inferred. The root locus method provides the roots of the system on the s plane, but then the results must be transformed to the time domain and the equation of the output determined. The transient response must be obtained by substituting values into this equation and obtaining the output at each specific time. Complicating this procedure is the fact that an infinite number of roots exist on the root locus, and enough of these roots must be considered to obtain reasonable accuracy. Each time an additional set of these complex conjugate roots is considered, additional time and complexity are added to the above procedure.

The transient response of this delayed second order system is shown in figure 5-5. It is here that the influence of the time delay on the second order system is most apparent. The original second order system was an overdamped system. It had no overshoot. The time delay has deteriorated the system to the point where the overshoot is approximately 70 percent.

It is interesting to note that a time delay of about 1.4 seconds would cause this system to be completely unstable.

The effect of the time delay on the transient response therefore is to increase overshoot and settling time. It can make a normally overdamped system highly oscillatory with a delay of only one second introduced.

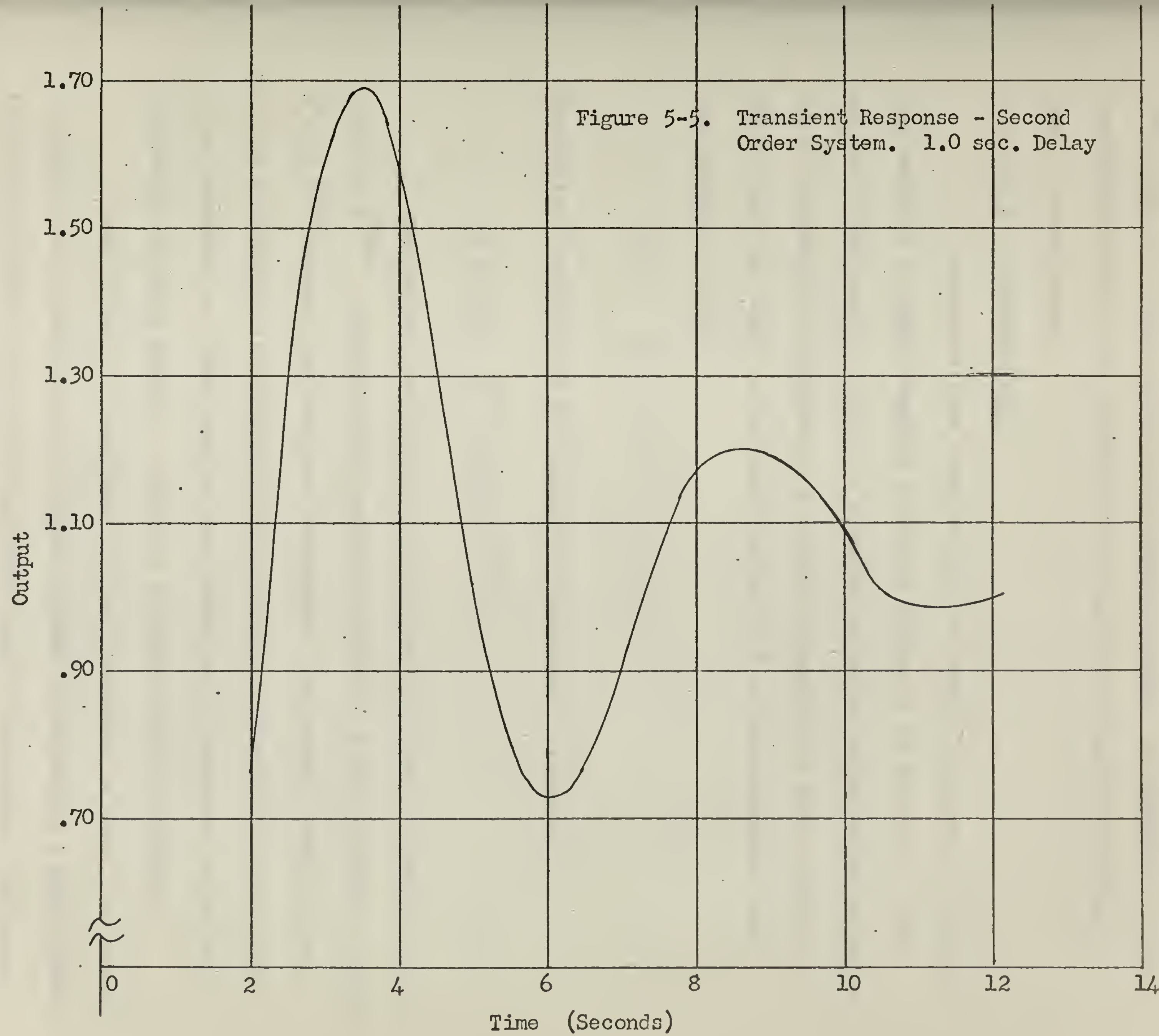


Figure 5-5. Transient Response - Second Order System. 1.0 sec. Delay

6. Transient Response by Numerical Methods

6.1 Discussion

A mathematical technique now exists for the direct solution of the output of a linear continuous feedback system. An extension of this numerical method to the solution of systems with transportation lag is easily accomplished.

6.2 The Z - Transformation

A continuous time function f(t) is sampled by means of a sampling switch S at equal sampling intervals of time T in figure 6-1. The output of the sampler f*(t) is a sequence of pulses which may be represented for mathematical purposes as a series of impulses or delta functions whose areas are equal to the amplitude of f(t) at the respective sampling instants. Thus

$$f^*(t) = \sum_{n=-\infty}^{+\infty} f(nT)\, \delta(t-nT) \qquad 6.1$$

The Laplace transform of the sequence of impulses is given as

$$\mathcal{L}[f^*(t)] = \sum_{n=0}^{+\infty} f(nT)\, e^{-nTs} \qquad 6.2$$

The Laplace transform of such pulsed functions, therefore, is in terms of e^{-nTs}. Replacing this exponential term by a new variable Z, where $Z^n = e^{nTs}$ accounts for the name Z-transform. One useful characteristic of the Z-transform is that the infinite series in equation 6.2 may usually be expressed in closed form so that tables of Z-transforms may be developed in the same manner as tables of ordinary Laplace transforms.

One of the big advantages of the Z-transform is that it is possible to obtain the pulsed output of a linear system by expressing a pulse transfer function which relates the input and output Z-transforms. The pulse

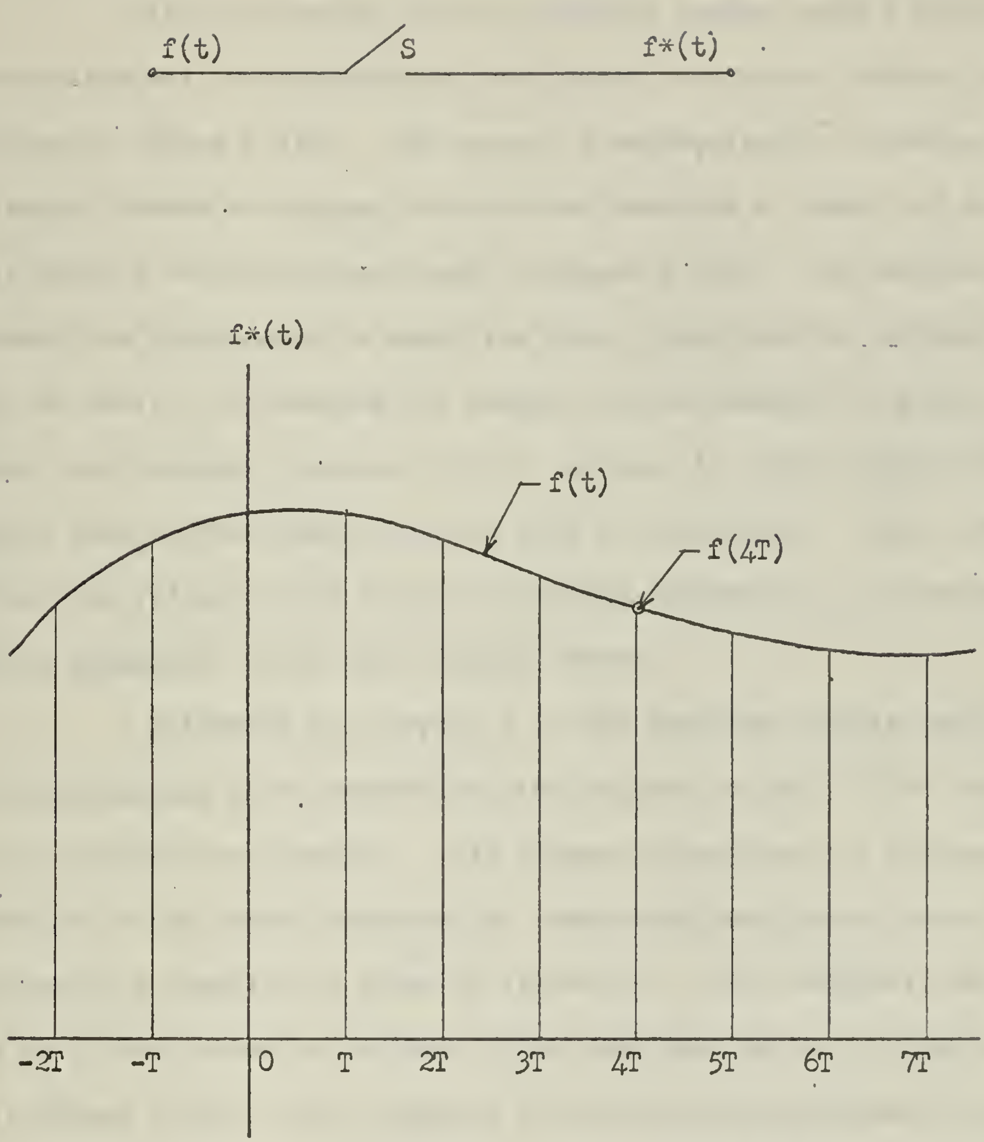


Figure 6-1. Sampled Time Function

transfer functions can be expressed in closed form and tables of Z-transforms are available in most standard texts covering sampled-data systems. An abbreviated table is given in Table 6-1.

6.3 Application to Linear Continuous Systems

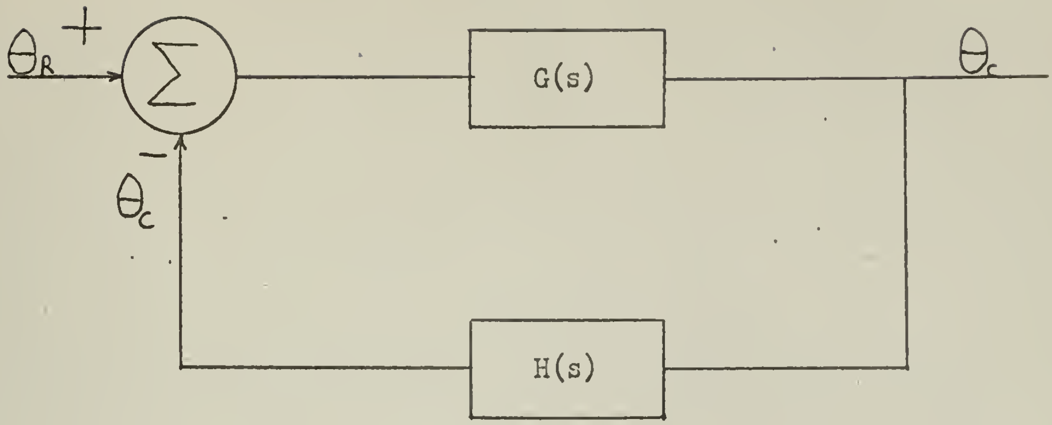
The application of the numerical method using Z-transform technique will be illustrated in a linear continuous feedback system as shown in figure 6-2(a). The system is mathematically converted into a sampled system by placing synchronized samplers at input and output and at point X in the feedback path in figure 6-2(b). The sampling frequency should be high enough to cause the error introduced by the sampled system to be small. By locating the sampler in the feedback loop and requiring that the transfer function G(s) be low pass in nature, then only the forward gain of the original system need be considered. Later examples will show the effect of the ratio of sampling frequency ω_s to the Bode cross-over frequency ω_c of the original system.

Following the sampler S in the feedback loop is the element B(s), whose function is to reconstruct the original output of the forward path as faithfully as possible. This element reconstructs a polygonal approximation to the actual function by converting each pulse into a triangular generating function as shown in figure 6-3. This requires the response of B to a short pulse or impulse of the same area to be a triangle as shown in figure 6-4(a). This response is not physically attainable since it requires a slope at negative time, but this has no effect on the purely mathematical solution.

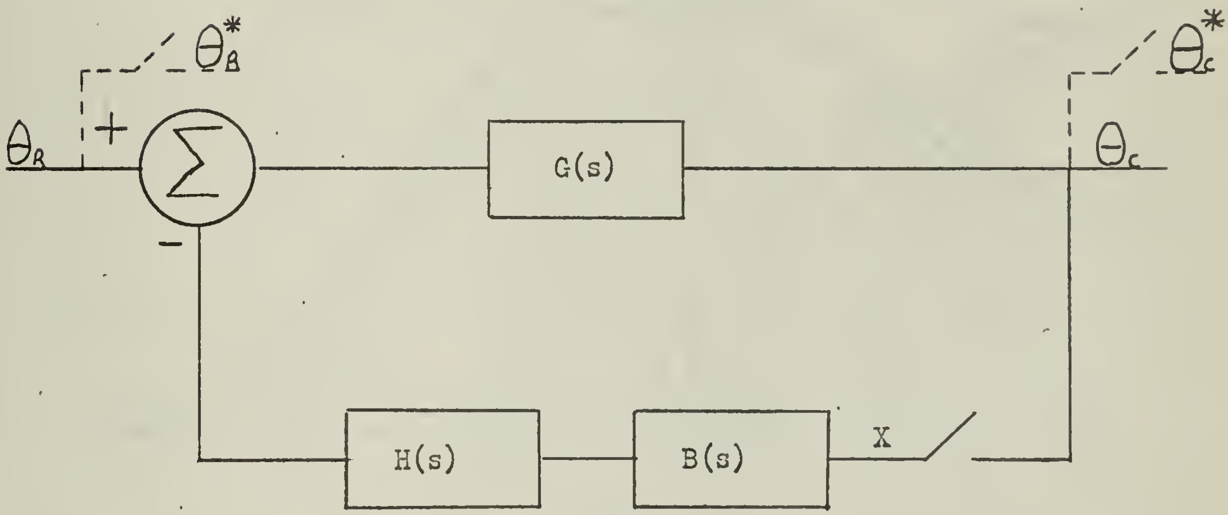
The combination of sampling switch operating with a period T and the triangle function generator B replace the direct connection which existed in the original system. The errors introduced into the

TABLE 6-1

Time Function $f(t)$	Laplace Transform $F(s)$	Z-Transform $F^*(z)$
$\delta(t-nT)$	e^{-nTs}	z^{-n}
$u(t)$	$\frac{1}{s}$	$\frac{1}{1-z^{-1}}$
t	$\frac{1}{s^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$
e^{-at}	$\frac{1}{s+a}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\frac{z^{-1}\sin aT}{1-2z^{-1}\cos aT+z^{-2}}$
$\cos at$	$\frac{s}{s^2+a^2}$	$\frac{1-z^{-1}\cos aT}{1-2z^{-1}\cos aT+z^{-2}}$



(a)



(b)

Figure 6-2. (a) Continuous Feedback System
(b) Sampled Model for Computation

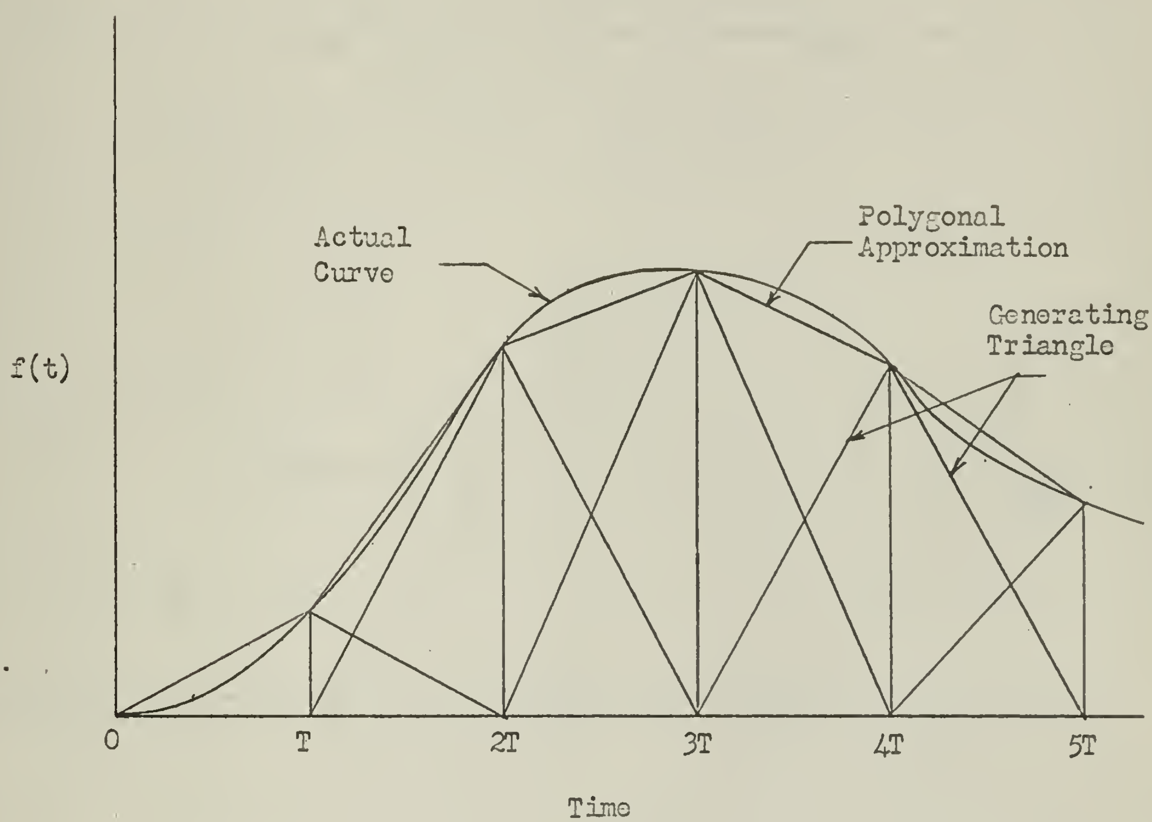


Figure 6-3. Polygonal Approximation of a Function f(t)

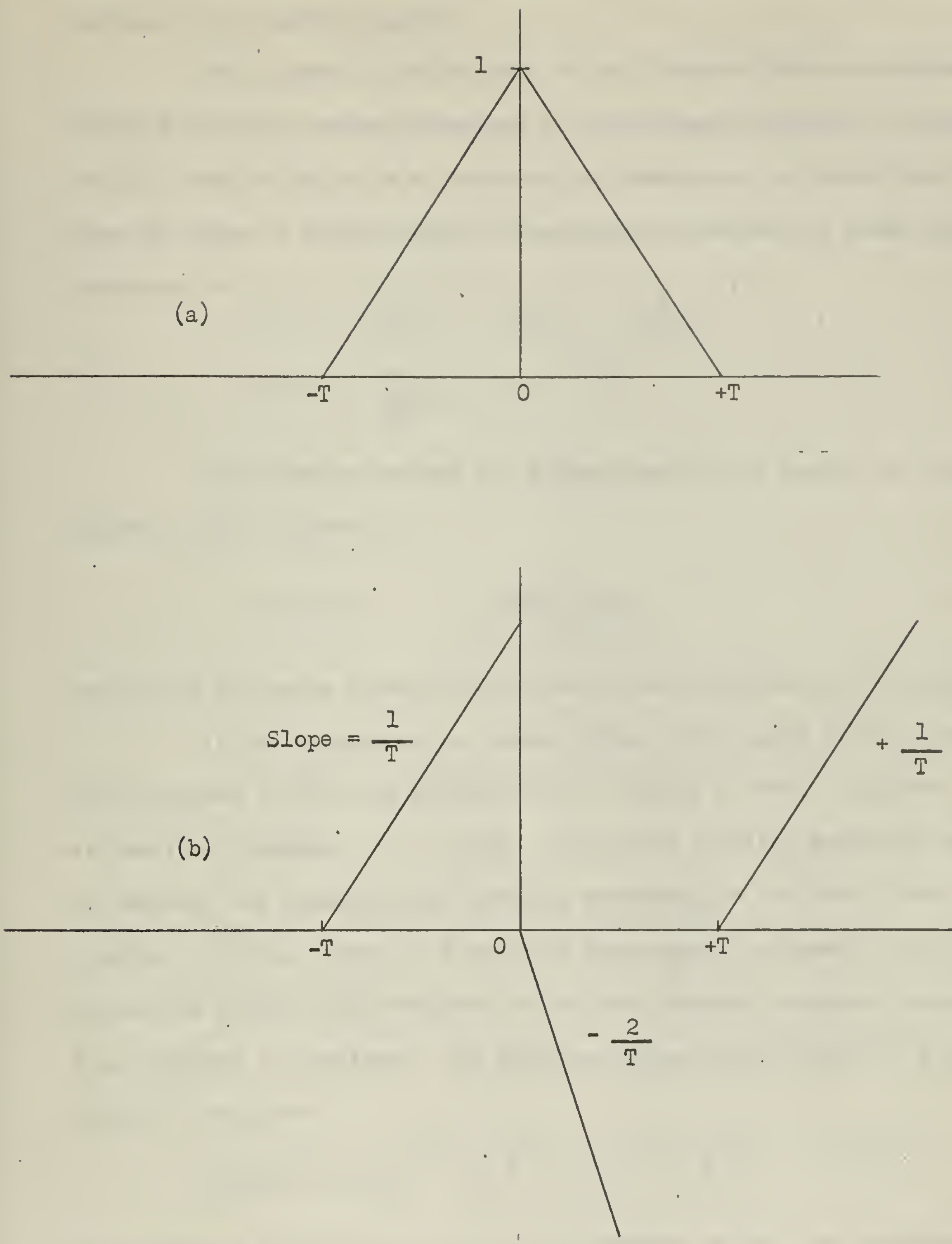


Figure 6-4. (a) Generating Triangle
(b) Ramps Which Produce Triangle

solution are a result of the inability of this combination to faithfully reproduce the feedback signal.

The transfer function B(s) of the triangle function generator is obtained from the Laplace transform of the triangle function in figure 6-4(a). This triangle is a result of the combination of three ramp functions as shown in figure 6-4(b). The Laplace transform of these three functions is:

$$B(s) = \frac{1}{Ts^2}e^{Ts} - \frac{2}{Ts^2} + \frac{1}{Ts^2}e^{-Ts}$$

or

$$B(s) = \frac{e^{Ts}}{Ts^2}\left(1 - e^{-Ts}\right)^2 \qquad 6.3$$

The relation between the Z-transform of the output and input in figure 6-2(b) is given by

$$\theta_c^*(z) = \frac{\theta_R G^*(z)}{1 + HBG^*(z)} \qquad 6.4$$

and yields the pulse transfer functions for which tables are available.

If the response of a linear system to an input F(s) is desired, first replace it with the system shown in figure 6-2(b). A proper choice of sampling frequency ω_s is made. The pulse transfer functions RG*(z) and HBG*(z) are computed, and combined according to the basic laws of algebra, yielding a form of a ratio of polynomials in powers of z. To obtain the output pulse sequence in the time domain, a simple process of long division is required. The division process will result in a power series of the form

$$\theta_c(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + \cdots + a_n z^{-n} + \cdots \qquad 6.5$$

The magnitude of the output in the time domain is just the coefficient of each of the terms in this series at a time corresponding to the power of z. Thus the output at the fourth sampling instant is a_4 in the expansion.

The long division process is continued until the desired number of points have been obtained. The effect of the sampling period T_s, or of the sampling frequency ω_s will be elaborated on in section 6.5.

6.4 Application to Linear Continuous Systems with Transportation Lag.

It has been shown in section 4.1 that the transfer function of a transportation lag or simple dead time lag is merely e^{-Ts} where T is the value T_D, or the time lag.

If this transfer function is inserted into the linear system of figure 6-2(b) as $G_1(s)$ then the solution for the output will become

$$\theta_C^*(z) = \frac{\theta_R G G_1^*(z)}{1 + HBGG_1^*(z)} \qquad 6.6$$

The value of this numerical method employing pulse transforms is now easily seen. The z-transform of the time delay $e^{-T_D s}$ simply becomes z^{-1} from the basic definition of the z-transform itself. Since the remaining terms of $\theta_C^*(z)$ are also polynomials in powers of z, this does not add any more complexity to the basic long division solution of the transient response.

It must be noted that the T introduced in the polynomial reconstruction and the z-transforms in Table 6-1 is T_s, the sampling interval. This must be distinguished and kept separate from T_D, the time delay.

One of the basic rules for the method is that the sampling time T_s must be equal to or an integral sub-multiple of the time delay T_D.
It will be necessary to express $\theta_C^*(z)$ only in terms of the sampling time T_s in order that the value of the transient response will be obtained at each sampling interval. This method will be shown in the sample solutions in section 7.

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

6.5 Requirements for Reasonable Accuracy

The ratio of ω_s to the crossover frequency ω_0 of the open loop transfer function will determine the relative accuracy of the system. The comparison of sampling period T_s with the expected settling time and/or time delay in the system will determine whether the method will supply the required answer within a reasonable number of long division steps.

Ideally, to obtain extremely good accuracy (within 1% of known answer) it is necessary to select a value of ω_s which is twice the frequency at which the open loop system is down 40 db on the Bode diagram. For the systems investigated here and explained in section 2, ω_0 is approximately 1 rad/sec. For the system where $G(s) = \frac{1}{s}$, this would require $\omega_s = 200$ rad/sec. Since $T_s = \frac{2\pi}{\omega_s}$ this would require a T_s of approximately 0.03 sec. With a normal settling time for this system of 4 seconds this would require a total of $\frac{4}{.03}$ or 133 long division steps to obtain the transient value at the end of 4 seconds. In order to reduce the computational work and to remain within approximately an accuracy of 5%, various ratios of ω_s to ω_0 were selected and results tabulated against results of former traditional methods of solution.

The problem of reducing computational work is further aggravated by the introduction of the time delay into the system. One basic requirement of this numerical method is that the sampling time T_s must be equal to or an integral sub-multiple of the time delay T_D. For systems with a long settling time, for example 5 seconds, and a short time delay of 0.10 seconds, it would require 50 long division steps to produce the value of the transient response at 5 seconds. This is only if it would be possible to select $T_s = T_D$. In the event that the value of T_s must be chosen smaller to satisfy the requirement of the ratio ω_s to ω_0 for

accuracy, then it would require 50n long division steps, where

$$n = \frac{T_D}{T_s}$$

It will be demonstrated in the next section that this method will provide reasonable engineering accuracy (approximately 5%) with a small investment of time. This can be accomplished with a ratio of ω_s to ω_0 as low as possible, and a T_s which is the lowest sub-multiple of T_D as long as it is not greater than $\frac{2\pi}{\omega_s}$.

This method can give extremely good accuracy at the expense of an extremely long and repetitious series of calculations. Its advantage lies in the fact that where the ratios mentioned above are favorable, it should be possible to compute the transient response of a system in a short period of time with no more powerful tools than algebra and a slide rule or desk calculator.

The next section will demonstrate the accuracy and simplicity of the method in solving simple feedback systems. Computational time and labor will be compared with other methods currently available to the engineer.

7. Comparison of Numerical Method with Other Methods of Solution

7.1 Numerical Method Solution

The numerical method solution to the second order system described in section 2 will now be demonstrated in detail. The system to be solved is shown in figure 6-2(b). The input is a unit step. The time delay will be one second. The sampling time interval will also be one second. The table of results in section 7.5 will also show the solution results for other sampling intervals.

The output $\theta_c(s)$ in the Laplace notation is

$$\theta_c(s) = \frac{\dfrac{10e^{-sT_D}}{s^2(s+10)}}{1 + \dfrac{10e^{-sT_D}B(s)}{s(s+10)}} \qquad 7.1$$

where

$$B(s) = \frac{e^{sT_s}}{T_s s^2}\left(1 - e^{-sT_s}\right)^2 \qquad 7.2$$

Combining

$$\theta_c(s) = \frac{\dfrac{10e^{-sT_s}}{s^2(s+10)}}{1 + \dfrac{10(1-e^{-sT_s})^2}{s^3(s+10)}} \qquad 7.3$$

It is important to note that the T in the time delay represents T_D, the delay time, whereas the T in the expression for B(s) and also in the Z transforms represents T_s, the sampling time. In the algebraic work in combining the expressions it will be necessary to express T_D in terms of T_s and obtain the output equation in terms of T_s only. In this example, since T_D equals T_s equals one second, it will not be necessary to account for this difference.

By expressing numerator and denominator separately in partial fraction expansions, and then transforming each separately by means of the transforms in table I of section 6 we obtain

$$\theta_c(z) = \frac{\dfrac{z^2(9+z^{-1})}{10(1-z^{-1})^2}}{\dfrac{200-118z^{-1}+116z^{-2}+2z^{-3}}{200(1-z^{-1})}} \qquad 7.4$$

Combining numerator and denominator results in

$$\theta_c(z) = \frac{180z^{-2}+20z^{-3}}{200-318z^{-1}+234z^{-2}-114z^{-3}-2z^{-4}} \qquad 7.5$$

By carrying out the long division indicated in equation 7.5 we obtain a power series in Z

$$\theta_c(z) = 0.9z^{-2}+1.531z^{-3}+1.385z^{-4}+0.91z^{-5}+\cdots \qquad 7.6$$

We may directly transform this expression into the time domain. It is not really necessary since the only effect on the output will be to express the series in terms of the sampling periods. In either case the coefficients are the same and are the value of the transient response at each sampling instant.

$$\theta_c(t) = 0.9(t-2T)+1.531(t-3T)+1.385(t-4T)+\cdots \qquad 7.7$$

The introduction of a different time delay or sampling time into the problem merely changes the algebraic relationship between T_D and T_s and once this accounting has been accomplished, the algebra and final solution work is similar to that shown here. No additional difficulty is introduced into the problem.

7.2 Power Series

7.21 First Order System

The solution of the transient response of a first order system with transportation lag by the employment of a power series expansion is

quite straight forward, although tedious and time consuming. The system under consideration is that shown in figure 7-1(a) where T is equal to one second.

The closed loop transfer function may be written

$$\frac{\theta_c}{\theta_R} = \frac{\frac{e^{-Ts}}{s}}{1 + \frac{e^{-Ts}}{s}} \tag{7.8}$$

For a unit step input

$$\theta_R = \frac{1}{s} \tag{7.9}$$

Then

$$\theta_c = \frac{\frac{e^{-Ts}}{s^2}}{1 + \frac{e^{-Ts}}{s}} = \frac{e^{-Ts}}{s^2}\left(1 + \frac{e^{-Ts}}{s}\right)^{-1} \tag{7.10}$$

The quantity $(1 + e^{-Ts})^{-1}$ may be expanded in a power series with the following result.

$$\theta_c = \frac{e^{-Ts}}{s^2}\left[1 - \frac{e^{-Ts}}{s} + \frac{e^{-2Ts}}{s^2} - \frac{e^{-3Ts}}{s^3} + \frac{e^{-4Ts}}{s^4} - \dots\right] \tag{7.11}$$

$$\theta_c = \frac{e^{-Ts}}{s^2} - \frac{e^{-2Ts}}{s^3} + \frac{e^{-3Ts}}{s^4} - \frac{e^{-4Ts}}{s^5} + \frac{e^{-5Ts}}{s^6} - \dots \tag{7.12}$$

Since T = 1

$$\theta_c = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^3} + \frac{e^{-3s}}{s^4} - \frac{e^{-4s}}{s^5} + \frac{e^{-5s}}{s^6} - \dots \tag{7.13}$$

Taking the inverse Laplace transform of each term yields

$$\begin{aligned}\theta_c(t) = {} & \mu(t-1)(t-1) - \mu(t-2)\frac{(t-2)^2}{2!} + \mu(t-3)\frac{(t-3)^3}{3!} \\ & - \mu(t-4)\frac{(t-4)^4}{4!} + \mu(t-5)\frac{(t-5)^5}{5!} - \dots\end{aligned} \tag{7.14}$$

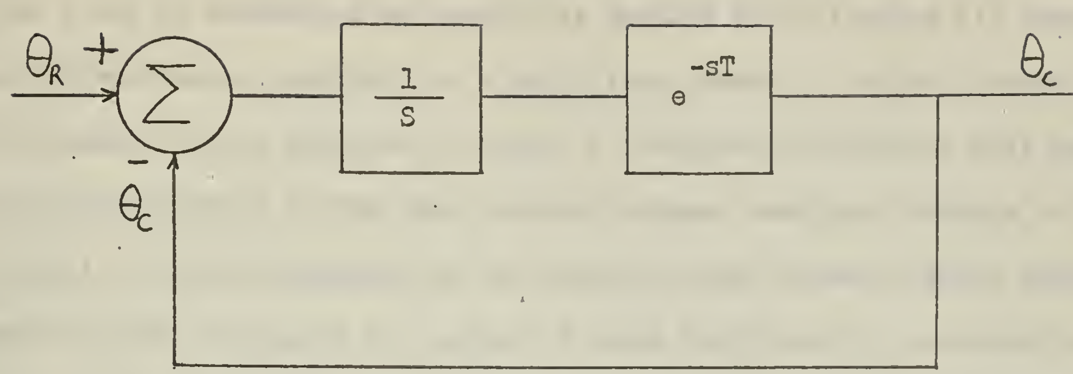


Figure 7-1(a). First Order Feedback Control System With Transportation Lag.

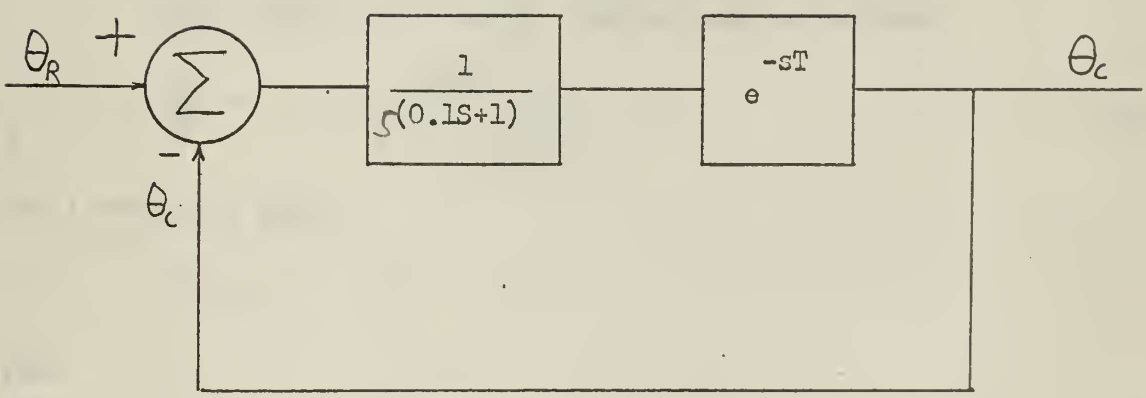


Figure 7-1(b). Second Order Feedback Control System With Transportation Lag.

The transient response may now be determined by substituting values of t into equation 7.14 . The magnitude of the output at any time t may be determined as exactly as desired by evaluating all terms and by evaluating each term to a sufficient number of decimal places. The amount of time required to obtain a transient solution by this method increases directly as the time interval between sampling instants is decreased. It also increases as the settling time increases since increased settling time increases the number of terms which must be evaluated at higher values of t. The sample problems solved by this method required about twelve hours to complete.

7.22 Second Order System

While the use of a power series expansion is a straight forward method of determining the transient response of a first order system with transportation lag, the method requires far too much work to be considered practical for higher order systems. The following example will demonstrate why the method fails as a practical working tool. The system under consideration is that shown in figure 7-1(b) where T is equal to one second.

The closed loop transfer function may be written

$$\frac{\theta_C}{\theta_R} = \frac{\dfrac{e^{-sT}}{s(0.1s+1)}}{1 + \dfrac{e^{-sT}}{s(0.1s+1)}} \qquad 7.15$$

For a unit step input

$$\theta_R = \frac{1}{s}$$

Then

$$\theta_C = \frac{\dfrac{e^{-sT}}{s^2(0.1s+1)}}{1 + \dfrac{e^{-sT}}{s(0.1s+1)}} = \frac{e^{-sT}}{s^2(0.1s+1)}\left(1 + \frac{e^{-sT}}{s(0.1s+1)}\right)^{-1} \qquad 7.16$$

The quantity $\left(1 + \dfrac{e^{-sT}}{s(0.1s+1)}\right)^{-1}$ may be expanded in a power series with the following results.

$$\theta_c = \frac{e^{-sT}}{s^2(0.1s+1)}\left[1 - \frac{e^{-sT}}{s(0.1s+1)} + \frac{e^{-2sT}}{s^2(0.1s+1)^2} - \frac{e^{-3sT}}{s^3(0.1s+1)^3} + \frac{e^{-4sT}}{s^4(0.1s+1)^4} - \cdots\right] \qquad 7.17$$

At this point it may readily be seen that the task of determining the inverse Laplace transform of each term is momentus and the work involved increases tremendously with each increase in the order of the term. Thus it is maintained that the use of a power series is impractical in determining the transient response of systems of second order or higher which are complicated by transportation lag.

7.3 Pade Approximation (Arithmetic)

The transient response will now be determined for the system of section 7.1 by simulating the Laplace shift operator with the second order Pade approximant. The major steps in the solution will be outlined but the tedious arithmetical calculations will of necessity not be repeated in their entirety.

The system in block diagram form is shown in Figure 7-1(b). Since the duration of the transportation lag is one second the Laplace shift operator becomes e^{-s}. As indicated in section 4.22

$$e^{-s} \cong \frac{s^2 - 6s + 12}{s^2 + 6s + 12} \qquad 7.18$$

Substituting the Pade approximant in for the Laplace shift operator and manipulating by block diagram algebra yields the closed loop transfer function for the system,

$$\frac{\theta_c}{\theta_R} = \frac{\dfrac{10(s^2-6s+12)}{s(s+10)(s^2+6s+12}}{1 + \dfrac{10(s^2-6s+12)}{s(s+10)(s^2+6s+12)}} \qquad 7.19$$

[illegible]

$$[illegible] \qquad (5.?)$$

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad (?)$$

[illegible]

$$[illegible] \qquad (?)$$

For a step input, $\theta_R = \frac{1}{S}$

$$\theta_C = \frac{1}{S}\left(\frac{10S^2 - 60S + 120}{S^4 + 16S^3 + 82S^2 + 60S + 120}\right) \qquad 7.20$$

The transient response of this system to a step input is contained in Laplace notation above. The problem of reverting to the time domain remains.

The first step is the determination of the four roots of $S^4 + 16S^3 + 82S^2 + 60S + 120$. Since these roots are two complex conjugate pairs whose real and imaginary parts are not necessarily integral the Lin method was selected for their solution. The first trial divisor is $82S^2 + 60S + 120$ which equals $S^2 + .733S + 1.46$. Five iterations were necessary to yield factors of $(S^2 + .463S + 1.65) (S^2 + 15.548S + 73.2)$. Application of the quadratic formula further reduced the factors to $(S + .232 \pm j1.264) (S + 7.77 \pm j3.59)$.

Next, the method of residues was employed to determine the coefficient of the inverse transform of each of the five poles.

Thus

$$\theta_C(t) = 1 + (-.312 \pm j.545)e^{(-.232 \pm j1.264)t} + (-.181 \pm j.238)e^{(-7.77 \pm j3.59)t} \qquad 7.21$$

This further reduces to

$$\theta_C(t) = 1 - .312e^{-.232t}(2\cos 1.264t + 3.46\sin 1.264t)$$
$$- .181e^{-7.77t}(2\cos 3.59t + 2.63\sin 3.59t) \qquad 7.22$$

The full transient response of the system may now be plotted by evaluating $\theta_C(t)$ at various values of t.

The time required for the solution of the transient response of the system by this method is between seven and eight hours. It is worthy of note that the roots of the characteristic equation determined by this method are reasonably close to the values obtained from the more accurate but more lengthy root locus solution. Arithmetic solutions making use of the Pade approximation were limited to the use of the second and third order approximants because the time required to solve a problem when higher order approximants are used becomes unreasonably long.

7.4 Pade Approximation (Computer)

In section 4.22 it was pointed out that the fourth order Pade approximant was more than adequate for handling the vast majority of problems involving transportation lag. Accordingly, all problems solved by arithmetical techniques were also solved on the Donner Analog Computer using the fourth order Pade approximant to simulate e^{-sT}. The solution to the problem by this method was established as the control against which the arithmetic solutions were compared. The fourth order approximant was chosen because it occupied only six amplifiers on the Donner Computer thus leaving four amplifiers free on which the remainder of the system could be programmed. The computer program for the fourth order Pade approximant is contained in the Appendix. This program was specifically written for a range of values of transportation lag between 0.1 seconds to 1.0 seconds. In each amplifier only one resistor R_T had to be changed to change the value of the transportation lag between the limits stated above. For transportation lags outside of these limits the program should be modified so that R_T does not exceed the bounds of 0.1 to 1.0 megohm.

7.5 Table of Results

This table of results is an intent to show the relative accuracy of the various methods, and to compare the solution time involved to achieve this accuracy. The first order system described in section 2 was used to establish the basic comparisons for accuracy, and to demonstrate the feasibility of the numerical method. Results indicate that the numerical method approach provides results of comparable accuracy to other methods with substantial savings in computational time. Results also indicate that increased accuracy may be obtained with the numerical methods solution by increasing the sampling frequency. It must be pointed out, however, that the time to arrive at the solution increases directly in proportion to the increase in the sampling frequency.

Table 7-1 contains the basic information from which general conclusions about relative accuracy will be based. The results indicate the value of the transient response for a first order system with a one second time delay as obtained by various methods. Results are shown at one second intervals. The power series solution is the most accurate available and will be taken as the standard by which the other methods will be compared.

Table 7-2 contains similar information for the first order system with a 0.5 second delay. The numerical method solution for a ratio of ω_s to ω_o ratio of 12.56 is compared because it achieves the stated requirements of the study of less than 5% deviation. It will be the ratio used in higher order systems.

Tables 7-3 and 7-4 display results for the second and third order systems respectively for a 0.5 second delay. The basis for comparison in

these systems was the computer solution employing the fourth order Pade approximant. The deviations here from the standard are of the same orders of magnitude as experienced in the first order system. In these results, a plus deviation indicates a result higher than the standard, and a minus deviation a result lower. The numerical method solution when compared with the standard characteristically maintains the same sign at the same relative point of the transient response. That is, the transient response as determined by the numerical method coincides with or leads the standard, and would therefore tend to show a slightly shorter rise time and time to peak overshoot. Its accuracy in determining the peak overshoot and settling time was well within the engineering accuracy prescribed for this investigation.

It must be understood that the solutions by the numerical methods technique yield "exact" values only at the sampling instants. Interpolation may be done to determine information at times between these sampling instants with assurance of reasonable accuracy as the magnitude and alternating signs of the deviations of the tabulated results from the standard indicate.

All solutions, both computer and analytic, employing the Pade approximation exhibit large deviations at the first few tabulated times. These are due to the "ringing" discussed in section 4.22 and illustrated in figure 4-3. This is an unavoidable disadvantage of the Pade approximation of e^{-sT}.

It cannot be over emphasized that the time required to obtain the transient solution by the numerical method was in most cases from one-fifth to one-tenth the time required by previously used analytical

and graphical methods. Its value also lies in the fact that the procedure is simple and easy to follow. Once the engineer or student becomes familiar with the procedures, it is possible to recognize and correct computational errors before they are compounded in following steps. This is simply not possible in the other analytical methods until the final step of the solution. Similarly, a change in the transportation lag or the sampling interval is easily handled, and a great deal of the algebraic manipulations need be done only once for each basic system.

TABLE 7-1

COMPARISON OF TRANSIENT RESPONSE AS OBTAINED BY

POWER SERIES WITH OTHER ARITHMETIC METHODS

FIRST ORDER SYSTEM - TRANSPORTATION LAG EQUAL TO ONE SECOND

Time Sec.	Power Series	2nd Order (Arith.)	% Dev.	Num. Meth. $\frac{\omega_s}{\omega_o}=6.28$	% Dev	Num. Meth. $\frac{\omega_s}{\omega_o}=12.56$	% Dev.	Num. Meth. $\frac{\omega_s}{\omega_o}=25.12$	% Dev.
0	0	.011	-	0	-	0	-	0	-
1	0	.291	-	0	-	0	-	0	-
2	1.00	1.013	+1.3	1.00	0	1.00	0	1.00	0
3	1.50	1.485	-1.0	1.50	0	1.50	0	1.50	0
4	1.167	1.158	-0.8	1.250	+7.1	1.1875	+1.8	1.1718	+0.41
5	.791	.795	+0.5	.875	+10.6	.8125	+2.7	.7969	+0.75
6	.8416	.848	+0.8	.8125	-3.5	.8320	-1.1	.8392	-0.28
7	1.057	1.055	-0.2	.9688	-8.3	1.0331	-2.3	1.0509	-0.58
8	1.100	1.098	-0.2	1.0782	-2.0	1.0993	-0.1	1.1022	+0.2
9	1.004	1.005	+0.1	1.0548	+4.9	1.0205	+1.6	1.0087	+0.37
10	.948	.950	+0.2	.9884	+4.2	.9564	+0.9	.9543	+0.66
11	.9798	.981	+0.1	.9669	-1.3	.9719	-0.8	.9916	+1.20
12	1.021	1.019	-0.2	.9894	-3.1	1.0108	-1.0	1.0329	+1.15

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TABLE 7-2

COMPARISON OF TRANSIENT RESPONSE AS OBTAINED

BY POWER SERIES WITH NUMERICAL METHOD RESULT

FIRST ORDER SYSTEM-TRANSPORTATION LAG EQUAL TO 0.5 SECOND

Time Sec	Power Series	Numerical Method $\frac{\omega_s}{\omega_o} = 12.56$	% Dev.
0	0	0	0
0.5	0	0	0
1.0	0.5	0.5	0
1.5	.875	.875	0
2.0	1.028	1.031	+0.29
2.5	1.039	1.055	+1.06
3.0	1.021	1.033	+1.18
3.5	1.006	1.011	+0.5
4.0	.999	1.000	+0.1

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TABLE 7-3

COMPARISON OF TRANSIENT RESPONSE FROM COMPUTER SOLUTION

WITH NUMERICAL METHOD AND PADE APPROXIMANT (ARITHMETIC)

SECOND ORDER SYSTEM - TRANSPORTATION LAG EQUAL TO 0.5 SECOND

Time Sec	Computer	Numerical Method $\frac{\omega_s}{\omega_o} = 12.56$	% Dev	Pade Approx. (Arith)	% Dev.
0	0	0	-	.01	-
0.5	0	0	-	.0018	-
1.0	.31	.35	+12.8	.404	+30.4
1.5	.77	.803	+ 4.3	.846	+ 9.9
2.0	1.04	1.078	+ 3.7	1.073	+ 3.2
2.5	1.115	1.153	+ 3.4	1.111	- 0.5
3.0	1.100	1.110	+ 0.9	1.066	- 3.1
3.5	1.050	1.037	- 1.2	1.017	- 3.1
4.0	1.015	.984	- 3.1	.992	- 2.3

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TABLE 7-4

COMPARISON OF TRANSIENT RESPONSE FROM COMPUTER SOLUTION

WITH NUMERICAL METHOD AND PADE APPROXIMANT (ARITHMETIC)

THIRD ORDER SYSTEM - TRANSPORTATION LAG EQUAL TO 0.5 SECOND

Time Sec.	Computer	Numerical Method $\frac{\omega_s}{\omega_o}=12.56$	% Dev.	Pade Approx. (Arith)	% Dev.
0	0	0	-	.146	-
0.5	0	0	-	.0203	-
1.0	.26	.35	+34.6	.359	+38.0
1.5	.73	.803	+10.0	.821	+12.5
2.0	1.050	1.078	+ 2.7	1.09	+ 3.8
2.5	1.150	1.153	+ 0.3	1.150	0
3.0	1.115	1.110	- 0.5	1.099	- 1.4
3.5	1.060	1.037	- 2.2	1.029	- 2.9
4.0	1.005	.984	- 2.1	.987	- 1.8

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8. Conclusions and Recommendations for Further Investigation

The investigation into the effect of transportation lag on feedback control systems reported on in this thesis has demonstrated four ideas of practical significance. It has also opened up a number of avenues for further investigation.

The primary conclusion which the investigation supported is the applicability of the numerical method technique devised by RAGAZZINI and BERGEN to the problem of determining the transient response of linear feedback control systems which are complicated by transportation lag. Of all the mathematical and graphical tools available to the engineer it offers the optimum combination of speed, accuracy, and simplicity. It may be used by students whose background includes courses only as advanced as linear feedback control systems, provided the Z-transform is included.

The deviation of the transient response as determined by the numerical method from that determined by the techniques currently in use may be held to less than three percent. Meanwhile the time required to determine the transient response has been reduced by more than seventy-five percent. Thus the decision of whether compensation is required or not may be made much more rapidly than before. The form the compensation will take will, of course, vary from problem to problem and from designer to designer.

The suitability of the numerical method to the particular problem of transportation lag suggests an area of further investigation. In determining the transient response the system has been mathematically converted to a sampled-data system, and is described by Z-transforms. In order to determine whether the system will meet specified performance

criteria it is necessary to calculate the transient response. If the system is not satisfactory, compensation techniques applicable to continuous feedback systems would be employed and the transient response of the compensated system recalculated. However, since the system is already a mathematical sampled-data system, it would appear that the suitability of sampled-data system performance criteria and compensation techniques to the problem is worthy of investigation. Should these techniques prove successful the reverting back and forth from the Laplace transform to the Z-transform for compensation could be avoided and a greater saving in time and tedium effected.

The second noteworthy conclusion of this investigation is that the Bode diagram not only offers the fastest tool for the determination of system stability but it also will indicate which set of roots will go unstable first. This information is made available by no other technique except the root locus and a conservative estimate would place the time differential using the root locus at a full order of magnitude higher than the Bode.

The criterion of dominance of a set of roots in a higher order system provides the basis for the extension of the performance characteristics of a second order system to these higher order systems. From the information in the Bode and root locus graphical plots, it may be practical to construct a set of second order curves with transportation lag which might have applicability to the analysis of higher order systems.

A third significant conclusion is that the Pade approximation for the Laplace shift operator is considerably more linear in its phase shift characteristics than the recent analog computer literature would indicate. An inspection of figure 4-4 will reveal that the Pade approximation provides a completely linear phase shift until that point where it

deviates from the ideal. Thus, the simulation of time delay on the analog computer requires merely the choice of a high enough order Pade approximant which will linearly phase shift the frequencies desired. It would appear that the curve fitting techniques currently in existence are not justified due to their increased complexity and lack of a simple repetitive scheme for programming.

A fourth interesting result of this study is the information on stability limits for the second order system. When plotted on logarithmic cross-section paper as shown in Figure 5-4, the non-dimensionalized gain vs. the time delay produced a straight line curve from which it was possible to write an equation describing the relationship between gain at stability limit and transportation lag for any second order system.

This result is obtainable from root locus or frequency response plots and is easily presented because of the ease in non-dimensionalizing a second order system. Here again, a further extension of this information would be possible for higher order systems. The method of displaying the information would depend on the way in which the additional time constants of the higher order systems were handled.

9. Appendix - Computer Program for Fourth Order Pade Approximant

The Pade approximation of e^x is discussed in Chapter 4. An analog computer program for the fourth order approximant which was used in the study is developed here.

The Pade approximation states that

$$e^{x} = \lim_{(u+v)\to\infty} \frac{F_{u,v}(x)}{G_{u,v}(x)} \qquad 9.1$$

where

$$F_{u,v}(x) = 1 + \frac{vx}{(u+v)\,1!} + \frac{v(v-1)\,x^2}{(u+v)(u+v-1)\,2!} + \cdots$$

$$\cdots + \frac{v(v-1)\cdots 2\cdot 1\, x^{v}}{(u+v)(u+v-1)\cdots(u+1)\,v!} \qquad 9.2$$

and

$$G_{u,v}(x) = 1 - \frac{ux}{(v+u)\,1!} + \frac{u(u-1)\,x^2}{(v+u)(v+u-1)\,2!} - \cdots$$

$$\cdots + (-1)^{u}\, \frac{u(u-1)\cdots 2\cdot 1\, x^{u}}{(v+u)(v+u-1)\cdots(v+1)\,u!} \qquad 9.3$$

Substituting u = v = 4, and x = -sT yields

$$e^{-sT} = \frac{(sT)^4 - 20(sT)^3 + 180(sT)^2 - 840(sT) + 1680}{(sT)^4 + 20(sT)^3 + 180(sT)^2 + 840(sT) + 1680} \qquad 9.4$$

$$e^{-sT} = \frac{1 - \frac{20}{sT} + \frac{180}{(sT)^2} - \frac{840}{(sT)^3} + \frac{1680}{(sT)^4}}{1 + \frac{20}{sT} + \frac{180}{(sT)^2} + \frac{840}{(sT)^3} + \frac{1680}{(sT)^4}} \qquad 9.5$$

As shown in Section 4.22

$$e^{-sT} = \frac{\mathcal{L}\{f(t-T)\}}{\mathcal{L}\{f(t)\}} \qquad 9.6$$

Since the differential operator and Laplace operator are interchangeable when initial conditions are zero, equations 9.5 and 9.6 may be combined as follows

$$\frac{f(t-T)}{f(t)} = \frac{1 - \frac{20}{DT} + \frac{180}{D^2T^2} - \frac{840}{D^3T^3} + \frac{1680}{D^4T^4}}{1 + \frac{20}{DT} + \frac{180}{D^2T^2} + \frac{840}{D^3T^3} + \frac{1680}{D^4T^4}} \qquad 9.7$$

The final form of equation 9.7 from which the analog computer program shown in figure 9-1 was developed is

$$\begin{aligned} f(t-T) = f(t) &- \left[\frac{20}{T} f(t) + \frac{20}{T} f(t-T)\right] \frac{1}{D} \\ &+ \left[\frac{180}{T^2} f(t) - \frac{180}{T^2} f(t-T)\right] \frac{1}{D^2} \\ &- \left[\frac{840}{T^3} f(t) + \frac{840}{T^3} f(t-T)\right] \frac{1}{D^3} \\ &+ \left[\frac{1680}{T^4} f(t) - \frac{1680}{T^4} f(t-T)\right] \frac{1}{D^4}. \end{aligned} \qquad 9.8$$

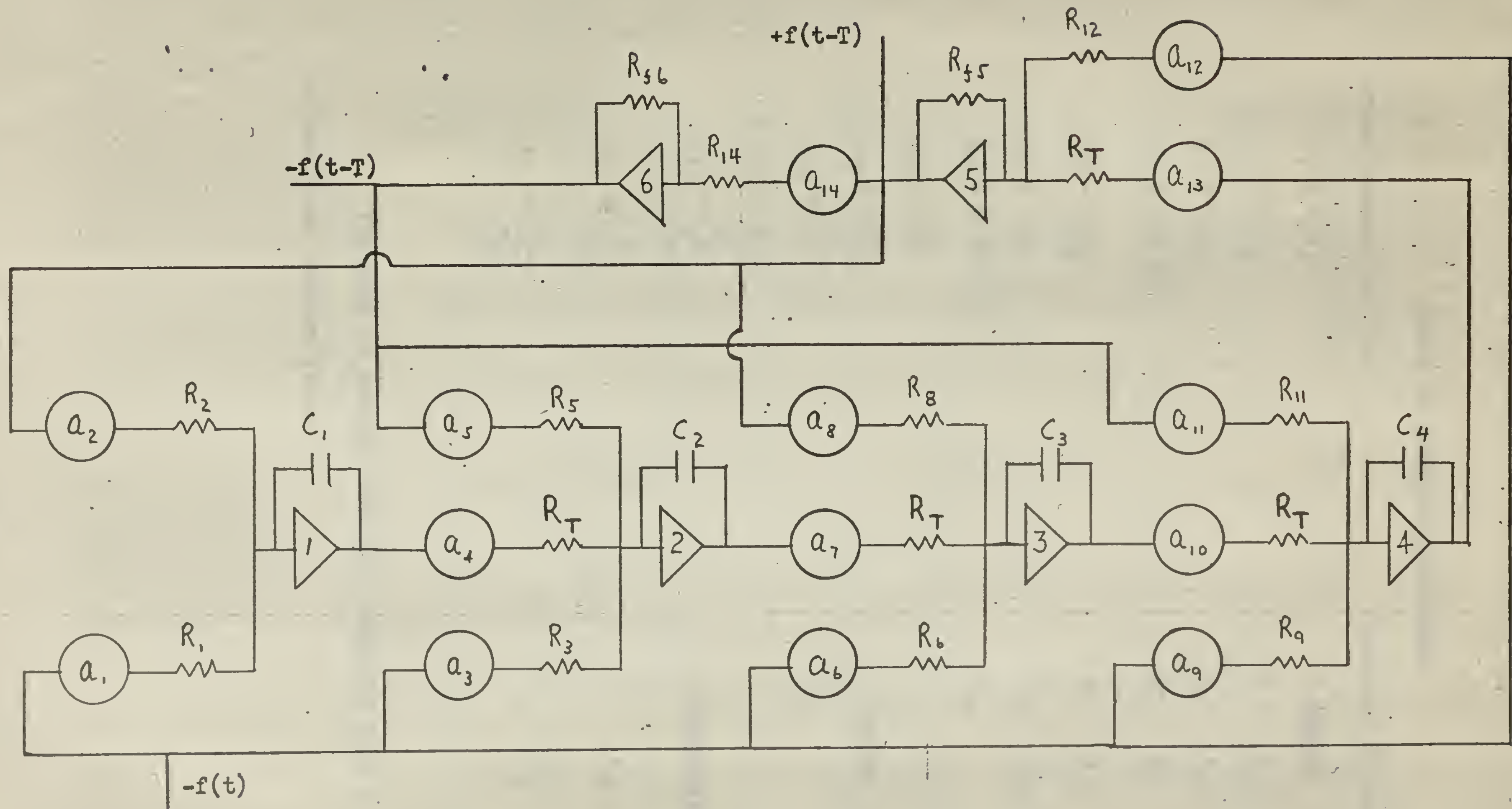

Figure 9-1. Analog Computer Program for Fourth Order Pade Approximant.

TABLE OF COMPUTER COMPONENTS

Potentiometer Settings	Component Values	
$a_1 = R_1C_1$	<u>1 Megohm</u>	
$a_2 = R_2C_1$	R_1	R_2
$a_3 = R_3C_2$	R_3	R_5
$a_4 = 0.2\ C_2$	R_6	R_8
$a_5 = R_5C_2$	R_9	R_{11}
$a_6 = R_6C_3$	R_{12}	R_{14}
$a_7 = 0.467\ C_3$	R_{f6}	
$a_8 = R_8C_3$		
$a_9 = R_9C_4$	<u>2 Megohm</u>	
$a_{10} = 0.9\ C_4$	R_{f5}	
$a_{11} = R_{11}C_4$		
$a_{12} = \frac{R_{12}}{R_{f5}}$	<u>1 Microfarad</u>	
$a_{13} = \frac{2}{R_{f5}}$	C_1	C_2
$a_{14} = \frac{R_{14}}{R_{f6}}$	C_3	C_4

The time is scaled such that computer time is equal to ten times problem time.

10. Bibliography

(1) RAGAZZINI, J. R. and BERGEN, A. R.: A Mathematical Technique for the Analysis of Linear Systems, Proc. IRE, Vol 42, no. 11, pp. 1645 1651, November 1954.

(2) RAGAZZINI, J. R. and ZADEH, L. H.: The Analysis of Sampled-Data Systems, Trans. AIEE, Vol 71, pt 2, pp. 225-234, November 1952.

(3) THALER, G. J. and BROWN, R. G.: Analysis and Design of Feedback Control Systems, McGraw-Hill Book Company, 1960.

(4) JOHNSON, C. L.: Analog Computer Techniques, McGraw-Hill Book Company, 1956.

(5) SMITH, G. W. and WOOD, R. C.: Principles of Analog Computation, McGraw-Hill Book Company, 1959.

(6) MORRILL, C. D.: A Sub-Audio Time Delay Circuit, IRE Trans. on Electronic Computers, Vol. EC-3, no. 2, pg. 45, June 1954.

(7) CHESTNUT, H. and MAYER, R. W.: Servomechanisms and Regulating System Design, John Wiley & Sons, 1959.

(8) TRUXAL, J. G.: Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, 1955.

(9) BOWER, J. L. and SCHULTHEISS, P. M.: Introduction to the Design of Servomechanisms, John Wiley & Sons, 1958.

(10) CHU, Y.: Synthesis of Feedback Control System by Phase Angle Loci, Trans. AIEE, pt 2, pp. 330-338, November 1952.

(11) TOU, J. T.: Digital and Sampled-Data Control Systems, McGraw-Hill Book Company, 1959.

(12) KUNZ, K. S.: Numerical Analysis, McGraw-Hill Book Company, 1957.